Math 709, 1-2

Note Title 13.02.2020

Yıldıray Ozan
ODTÜ Matematik Bölümü

Math 709 - General Topology

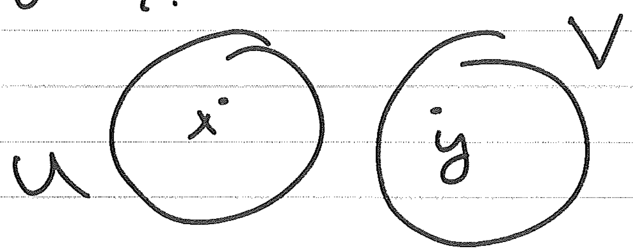
- Differentiable Manifolds
- Intersection Theory
- Vector Bundles
- Characteristic Classes and some applications.

Book: Türevlenebilir manifoldlara giriş
ODTÜ YAYINEVİ

Differentiable Manifolds

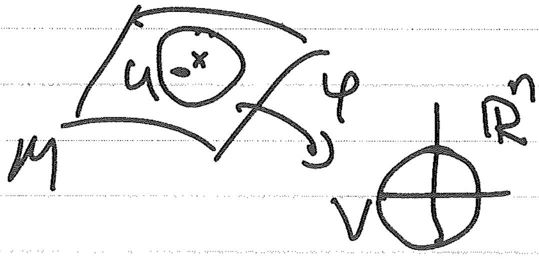
Definition: A topological manifold is a Hausdorff and second countable topological space, which is locally Euclidean.

Hausdorff $x, y \in X$, $x \neq y \Rightarrow \exists\, U, V \subseteq X$ open subsets such that $x \in U$, $y \in V$ and $U \cap V = \phi$.



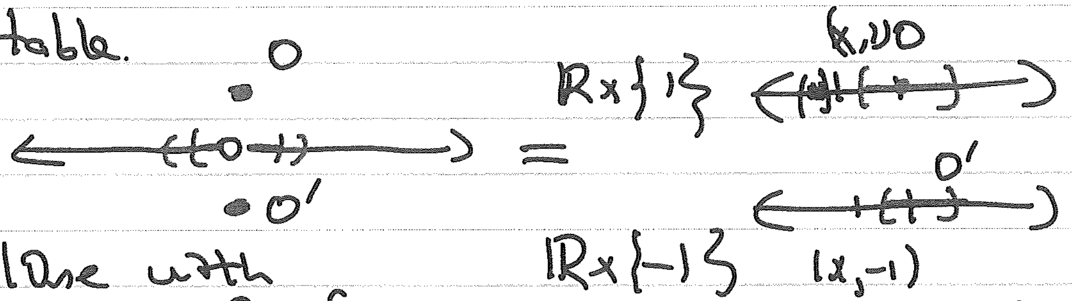
Second Countable: X has a countable basis $\mathcal{B}$.

Locally Euclidean: $x \in X$, $\varphi : U \to V$ homeomorphism s.t. $x \in U \subseteq X$ and $V \subseteq \mathbb{R}^n$ open subset.



__Remark:__ In order to embedd a manifold into some Euclidean space we must require that it is Hausdorff and second countable.

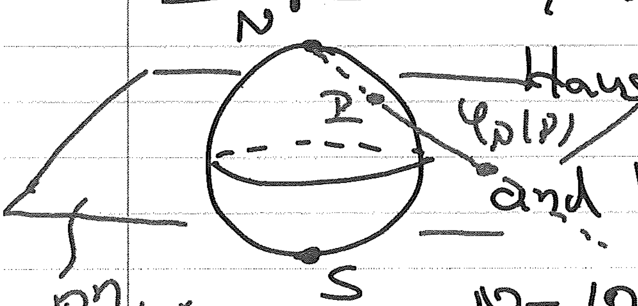
__Ex:__



The line with double origin is locally Euclidean but __not__ Hausdorff.

$(x,1)\sim(x,-1)$ if $x\neq 0$

__Example__ $S^n=\{(x_1,\dots,x_{n+1})\in\mathbb{R}^{n+1} \mid x_1^2+\dots+x_{n+1}^2=1\}$



Hausdorff, Second Countable and locally Euclidean.

$\mathbb{R}^n: x_{n+1}=0$ $\quad N=(0,\dots,0,1),\ S=(0,\dots,0,-1)$.

$U_N=S^n\setminus\{N\},\ U_S=S^n\setminus\{S\}$.

$\varphi_N: U_N\to\mathbb{R}^n$,

$$\varphi_N(x_1,\dots,x_{n+1})=\left(\frac{x_1}{1-x_{n+1}},\dots,\frac{x_n}{1-x_{n+1}}\right)$$

$\varphi_S : U_S \longrightarrow \mathbb{R}^n$

$\varphi_S(x_1, \dots, x_{n+1}) = \left(\frac{x_1}{1+x_{n+1}}, \dots, \frac{x_n}{1+x_{n+1}}\right)$

φ_N and φ_S are both homeomorphism with inverses

$\varphi_N^{-1} : \mathbb{R}^n \longrightarrow U_N,$

$\varphi_N^{-1}(y_1, \dots, y_n) = \left(\frac{2y_1}{1+\|y\|^2}, \dots, \frac{2y_n}{1+\|y\|^2}, \frac{\|y\|^2-1}{1+\|y\|^2}\right)$

$\|y\|^2 = y_1^2 + \dots + y_n^2,$ and

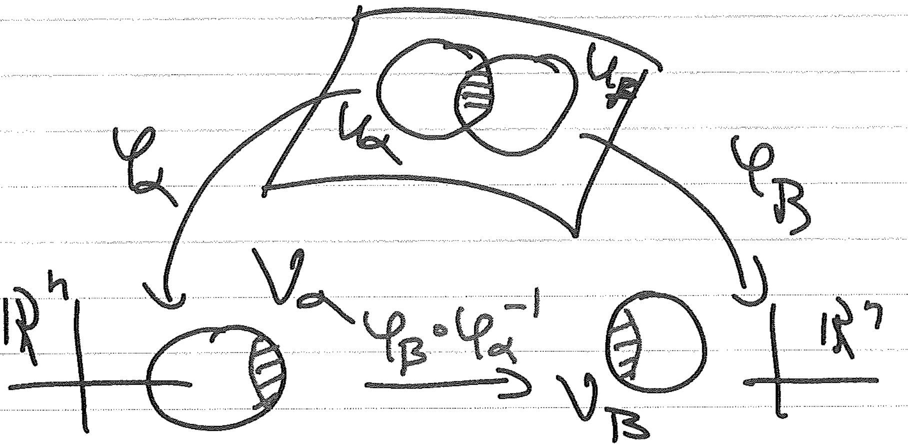
$\varphi_S^{-1} : \mathbb{R}^n \longrightarrow U_S,$

$\varphi_S^{-1}(y_1, \dots, y_n) = \left(\frac{2y_1}{1+\|y\|^2}, \dots, \frac{2y_n}{1+\|y\|^2}, \frac{1-\|y\|^2}{1+\|y\|^2}\right)$

$\Rightarrow S^n$ is locally Euclidean.

<u>Definition</u>: Let M be a topological manifold with an atlas $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha \in \Lambda}$, when $U_\alpha \subseteq M$ open, $V_\alpha \subseteq \mathbb{R}^n$ open, $\varphi_\alpha : U_\alpha \to V_\alpha$ homeomorphism, for each $\alpha \in \Lambda$ and $M = \bigcup_{\alpha \in \Lambda} U_\alpha$. If all compositions $\varphi_\beta \circ \varphi_\alpha^{-1}$, wherever they are defined, are

smooth maps of open subsets of Euclidean spaces then we say that the atlas $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha\in\Lambda}$ defines a smooth manifold structure on M.



$\varphi_\beta \circ \varphi_\alpha^{-1} \in C^\infty$.

<u>Back to the Example:</u>

$\varphi_N \circ \varphi_S^{-1} : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$

$$(y_1, \dots, y_n) \overset{\varphi_S^{-1}}{\longmapsto} \Big(\underbrace{\frac{2y_1}{1+\|y\|^2}}_{x_1}, \dots, \underbrace{\frac{2y_n}{1+\|y\|^2}}_{x_n}, \underbrace{\frac{1-\|y\|^2}{1+\|y\|^2}}_{x_{n+1}}\Big)$$

$$\Big\downarrow \varphi_N$$

$$\left(\frac{x_1}{1-x_{n+1}}, \dots, \frac{x_n}{1-x_{n+1}}\right)$$

$$\frac{x_1}{1-x_{n+1}} = \frac{2y_1 / 1+\|y\|^2}{1-\left(\frac{1-\|y\|^2}{1+\|y\|^2}\right)} = \frac{2y_1}{2\|y\|^2} = \frac{y_1}{\|y\|^2}$$

So, $(\varphi_N \circ \varphi_S^{-1})(y_1, \dots, y_n) = \frac{1}{\|y\|^2}(y_1, \dots, y_n)$ is C^∞ on $\mathbb{R}^n \setminus \{0\}$. Thus S^n is a smooth manifold of dimension n.

<u>Ex</u>: $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$

$\varphi_N : S^1 \setminus \{N\} \longrightarrow \mathbb{R}$, $(x,y) \longmapsto \frac{x}{1-y}$

$\varphi_S : S^1 \setminus \{S\} \longrightarrow \mathbb{R}$, $(x,y) \longmapsto \frac{x}{1+y}$

$\varphi_N^{-1} : \mathbb{R} \longrightarrow S^1 \setminus \{N\}$, $t \longmapsto \left(\frac{2t}{1+t^2}, \frac{t^2-1}{1+t^2}\right)$

$\varphi_S^{-1} : \mathbb{R} \longrightarrow S^1 \setminus \{S\}$, $t \longmapsto \left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)$.

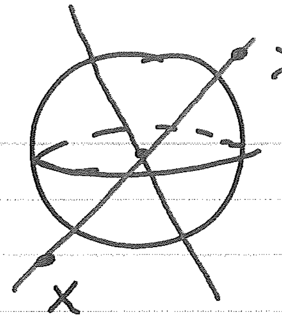
Moreover,

$\varphi_N \circ \varphi_S^{-1} : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R} \setminus \{0\}$

$$t \longmapsto \left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \longmapsto \frac{\frac{2t}{1+t^2}}{1-\frac{1-t^2}{1+t^2}} = \frac{2t}{2t^2} = \frac{1}{t}$$

<u>Example $\mathbb{RP}^n$</u>: the real projective space of dimension n.

$\mathbb{RP}^n$ = the space of lines in $\mathbb{R}^{n+1}$ through the origin



$$\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\} \Big/ \begin{matrix} x \sim \lambda x \\ x \in \mathbb{R}^{n+1} \setminus \{0\} \\ \lambda \in \mathbb{R} \setminus \{0\}. \end{matrix}$$

$\mathbb{RP}^n$ is second countable since $\mathbb{R}^{n+1}$ is second countable

<u>Exercise:</u> $\mathbb{RP}^n$ is Hausdorff.

$[x_0 : x_1 : \quad : x_n] = \{\lambda (x_0, x_1, \dots, x_n) \mid \lambda \neq 0\}$, the equivalence class containing the point $(x_0, x_1, \dots, x_n)$.

$U_i = \{[x_0 : x_1 : \quad : x_n] \mid x_i \neq 0\}$, $i = 0, \dots, n$.

$\varphi_i : U_i \longrightarrow \mathbb{R}^n$,

$$[x_0 : \quad : x_n] \longmapsto \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \widehat{\frac{x_i}{x_i}}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right)$$

$\varphi_i^{-1} : \mathbb{R}^n \to U_i$, $(y_1, \dots, y_n) \mapsto [y_1, \dots, y_{i-1}, 1, y_{i+1}, \dots, y_n]$.

φ_i is a homeomorphism for each i.

Hence, $\mathbb{RP}^n$ is a topological manifold of dimension n.

$$\varphi_j \circ \varphi_i^{-1} : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$$

$$(y_1, \dots, y_n) \underset{\varphi_i^{-1}}{\longmapsto} [y_1 : \quad : y_j : \quad : y_{i-1} : 1 : y_i : \dots : y_n)$$

$\varphi_j \downarrow$

$$\left(\frac{y_1}{y_j}, \dots, \widehat{\frac{y_j}{y_j}}, \dots, \frac{y_{i-1}}{y_j}, \frac{1}{y_j}, \frac{y_{i+1}}{y_j}, \dots, \frac{y_n}{y_j}\right)$$

are clearly smooth and thus the atlas $\{\varphi_i\}_{i=0}^{n}$ defines a smooth structure on $\mathbb{RP}^n$.

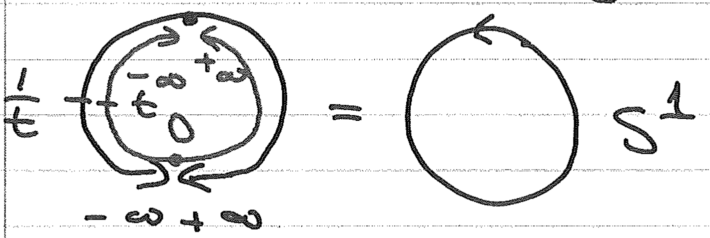
<u>Ex</u> $\mathbb{RP}^1 = U_0 \cup U_1$, $U_0 = \{[x_0 : x_1] \mid x_0 \neq 0\}$

$U_1 = \{[x_0 : x_1] \mid x_1 \neq 0\}$

$U_0 \longrightarrow \mathbb{R}$, $[x_0 : x_1] \longmapsto \overset{t}{x_1/x_0} = t$

$U_1 \longrightarrow \mathbb{R}$, $[x_0 : x_1] \longmapsto x_0/x_1 = 1/t$

$\mathbb{RP}^1 = \mathbb{R} \underset{0}{\dot\cup} \mathbb{R} / t \sim \frac{1}{t}$, $t \neq 0$.



<u>Complex Projective Space $\mathbb{CP}^n$</u>:

$\mathbb{CP}^n = \mathbb{C}^{n+1} \setminus \{0\} / (z_0, \dots, z_n) \sim \lambda(z_0, \dots, z_n)$

$\lambda \in \mathbb{C}, \lambda \neq 0.$

$\mathbb{C} = \mathbb{R}^2$, $\mathbb{C}^{n+1} = \mathbb{R}^{2n+1}$

$\mathbb{CP}^n$ Hausdorff and second countable.

$U_i = \{[z_0 : z_1 : \cdots : z_n] \mid z_i \neq 0\} \subseteq \mathbb{CP}^n$ open

$\varphi_i : U_i \longrightarrow \mathbb{C}^n = \mathbb{R}^{2n}$,

$\varphi_i([z_0 : \cdots : z_n]) = \left(\frac{z_0}{z_i}, \ldots, \widehat{\frac{z_i}{z_i}}, \ldots, \frac{z_n}{z_i}\right)$

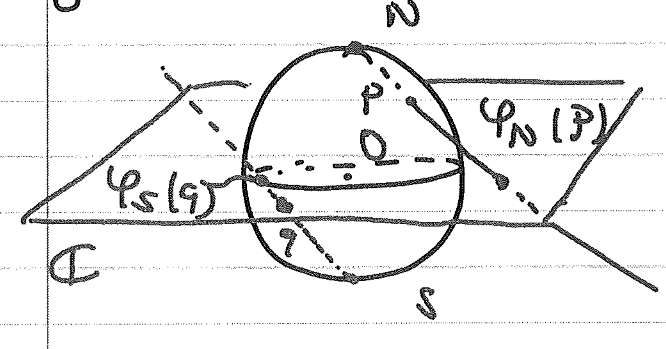
$\varphi_i^{-1} : \mathbb{C}^n \longrightarrow U_i$

$\varphi_i^{-1}(w_1, \ldots, w_n) = [w_1 : \ \ : w_{i-1} : 1 : w_i : \cdots : w_n]$

$\varphi_j \circ \varphi_i^{-1}$ is a C^∞ function of open subsets of $\mathbb{C}^n = \mathbb{R}^{2n}$, and thus $\mathbb{CP}^n$ is a smooth 2n-dimensional manifold.

Ex: $\mathbb{CP}^1 = \mathbb{C} \dot\cup \mathbb{C} \Big/ z \sim \frac{1}{z},\ z \neq 0$

This description shows that $\mathbb{CP}^1$ is just the Riemann Sphere.



$\varphi_N \circ \varphi_S^{-1} : \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C} \setminus \{0\}$

$z \longmapsto z^{-1}$

Example: $\mathbb{HP}^n$: Quaternionic projective n space

$\mathbb{H} = \mathbb{R}^4 \quad (x_1, x_2, x_3, x_4) \longleftrightarrow x_1 + i x_2 + j x_3 + k x_4$

$i^2 = j^2 = k^2 = -1, \quad i \cdot j = k, \quad j \cdot k = i, \quad k \cdot i = j$

$j \cdot i = -k, \quad k \cdot j = -i, \quad i \cdot k = -j$

$\mathbb{HP}^n = \mathbb{H}^{n+1} \setminus \{0\} / v \sim \lambda v, \quad \lambda \in \mathbb{H} \setminus \{0\}$.

$\mathbb{HP}^1 = \mathbb{H} \cup \mathbb{H} / p \sim \frac{1}{p}, \; p \neq 0$.

Remark: $\mathbb{RP}^1 = S^1, \; \mathbb{CP}^1 = S^2, \; \mathbb{HP}^1 = S^4$

$\mathbb{RP}^1 \downarrow w_i$ Stiefel Whitney; $\mathbb{CP}^1 \downarrow c_i$ Chern Classes; $\mathbb{HP}^1 \downarrow$ Pontryagin classes

Tangent Space and Tangent Bundle

$p \in \mathbb{R}^n$, $T_p\mathbb{R}^n$ = The set of all derivations on the ring of smooth functions defined near $p \in \mathbb{R}^n$.

$v_p(f) \in \mathbb{R}, \quad f : U \to \mathbb{R}$ smooth, $p \in U \subseteq \mathbb{R}^n$

- v_p linear : $v_p(af + bg) = a\, v_p(f) + b\, v_p(g)$
  $a, b \in \mathbb{R}$

- $v_p(fg) = v_p(f)\, g(p) + f(p)\, v_p(g)$

__Ex__ $\mathbb{R}^n$, $x_1, \dots, x_n$ coordinate on $\mathbb{R}^n$

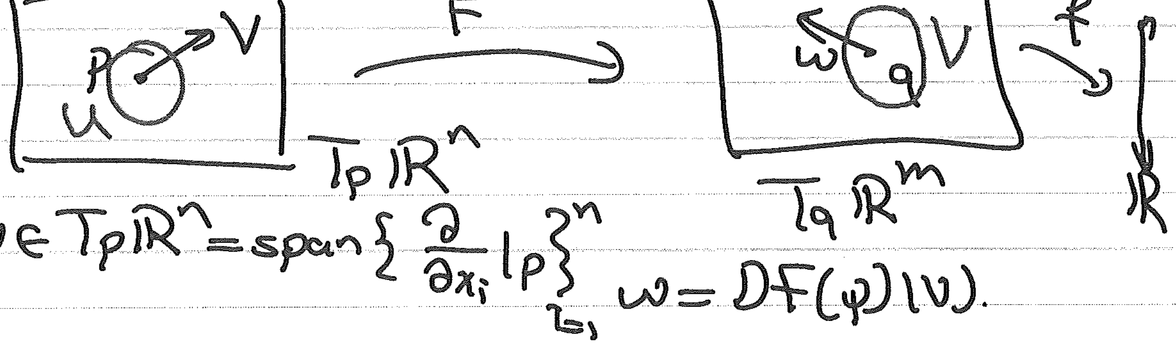
$\left(\frac{\partial}{\partial x_i}\Big|_p\right)(f) = \frac{\partial f}{\partial x_i}(p)$ is a derivation

__Proposition__ The set of all derivations $T_p\mathbb{R}^n$ is a vector space of dimension n and $\left\{\frac{\partial}{\partial x_i}\Big|_p\right\}_{i=1}^n$ is a basis for $T_p\mathbb{R}^n$.

— o —

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth function. Then for any $p \in \mathbb{R}^n$ we have a linear map

$$DF(p) : T_p\mathbb{R}^n \longrightarrow T_q\mathbb{R}^m, \quad q = F(p)$$



$v \in T_p\mathbb{R}^n = \text{span}\left\{\frac{\partial}{\partial x_i}\Big|_p\right\}_{i=1}^n$, $w = DF(p)(v)$.

$w \in T_q\mathbb{R}^m$, $f : V \to \mathbb{R}$, $q \in V \subseteq \mathbb{R}^m$ open

$U = F^{-1}(V)$. So $f \circ F : U \to \mathbb{R}$ smooth function.

In this case $DF(p)(v)$ is defined to be the derivation given by

$$DF(p)(v)(f) = v(f\circ F), \quad v = \frac{\partial}{\partial x_i}\Big|_p$$

$$v(f\circ F) = \frac{\partial}{\partial x_i}(f\circ F)(p) \qquad F = (f_1, \dots, f_m)$$

$$f_i : \mathbb{R}^n \to \mathbb{R}$$

$$= \frac{\partial}{\partial x_i}\left(f(f_1, f_2, \dots, f_m)\right)(p) \qquad q = F(p)$$

$$= \frac{\partial f}{\partial y_1}(q)\frac{\partial f_1}{\partial x_i}(p) + \frac{\partial f}{\partial y_2}(q)\frac{\partial f_2}{\partial x_i}(p) + \cdots + \frac{\partial f}{\partial y_m}(q)\frac{\partial f_m}{\partial x_i}(p)$$

$$= \begin{bmatrix} \frac{\partial f_1}{\partial x_i} & \frac{\partial f_2}{\partial x_i} & \cdots & \frac{\partial f_m}{\partial x_i} \end{bmatrix}(p) \begin{bmatrix} \frac{\partial f}{\partial y_1} \\ \frac{\partial f}{\partial y_2} \\ \vdots \\ \frac{\partial f}{\partial y_m} \end{bmatrix}(q)$$

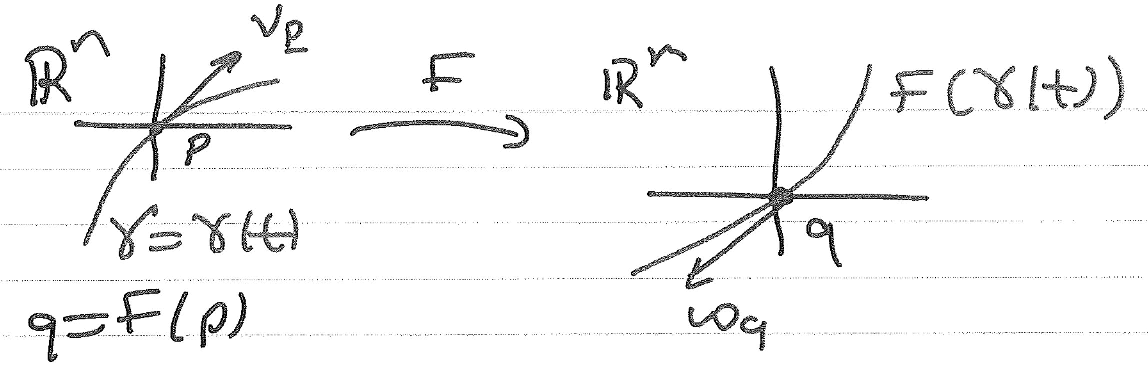
$$T_p\mathbb{R}^n = \mathrm{span}\left\{\frac{\partial}{\partial x_1}\Big|_p, \dots, \frac{\partial}{\partial x_n}\Big|_p\right\} \leftarrow \beta$$

$$T_q\mathbb{R}^m = \mathrm{span}\left\{\frac{\partial}{\partial y_1}\Big|_q, \dots, \frac{\partial}{\partial y_m}\Big|_q\right\} \leftarrow \beta'$$

$$DF(p) : T_p\mathbb{R}^n \to T_q\mathbb{R}^m$$

$$[DF(p)]_{\beta}^{\beta'} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}(p) = \frac{\partial(f_1, \dots, f_m)}{\partial(x_1, \dots, x_n)} = \frac{\partial F}{\partial(x_1, \dots, x_n)}(p)$$

Jacobian of F at $p \in \mathbb{R}^n$.



$q = F(p)$

$$w_q = \frac{d}{dt}\left(F(\gamma(t))\right)\Big|_{t=0}, \quad \frac{d\gamma}{dt}(0) = v_p$$

$$= DF(p)(v_p).$$

Note Title

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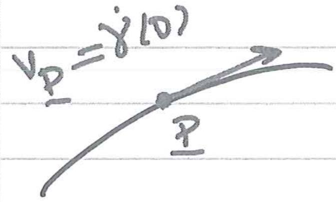
$\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $v_p \in T_p\mathbb{R}^n$

$D\Phi_p(v_p) = ?$

$D\Phi_p(v_p) = \frac{d}{dt}\Phi(\gamma(t))\Big|_{t=0}$, $\gamma : (-\epsilon,\epsilon) \to \mathbb{R}$

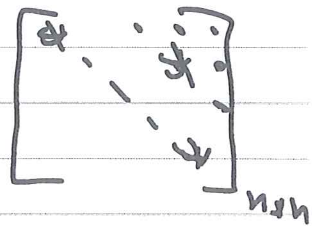
$\gamma(0) = p$, $\dot\gamma(0) = v_p$



Ex $\Phi : M(n) \longrightarrow S(n)$

$1+2+\cdots+n = \frac{n(n+1)}{2}$

$M(n) = n\times n$ - Real matrices
$= \mathbb{R}^{n\times n} \simeq \mathbb{R}^{n^2}$



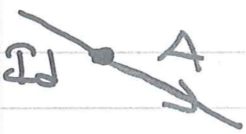
$S(n) = n\times n$ - Symmetric matrices
$= \{[a_{ij}] \in M(n) \mid a_{ij} = a_{ji}\}$
$= \mathbb{R}^{n(n+1)/2}$

$Id \in M(n)$, $D\Phi_{Id}(A) = ?$

$T_{Id}M(n) = M(n)$, $T_{Id}S(n) = S(n)$

$\Phi : M(n) \longrightarrow S(n)$, $\Phi(Q) = Q^TQ$

$D\Phi_{Id}(A) = \frac{d}{dt}\Phi(\gamma(t))\Big|_{t=0}$



$\gamma(t) = Id + tA$

$$D\hat{\Phi}_{Id}(A) = \frac{d}{dt}\left(\gamma(t)^T\gamma(t)\right)\Big|_{t=0}$$

$$= \frac{d}{dt}\left[(Id + tA)^T(Id + tA)\right]\Big|_{t=0}$$

$$= \frac{d}{dt}\left(Id + tA^T + tA + t^2A^TA\right)\Big|_{t=0}$$

$$= \left(0 + A^T + A + 2tA^TA\right)\Big|_{t=0}$$

$$= A^T + A.$$

So $D\hat{\Phi}_{Id}(A) = A^T + A$.

<u>Claim:</u> $D\hat{\Phi}_{Id} : T_{Id}M(n) \to T_{Id}S(n)$ is onto.

<u>Proof:</u> If $B \in T_{Id}S(n) = S(n)$, then $B = B^T$,

$$B = \frac{B}{2} + \frac{B^T}{2} = D\hat{\Phi}_{Id}(A), \text{ where } A = \frac{B}{2}.$$

<u>Tangent Space of Manifolds</u>

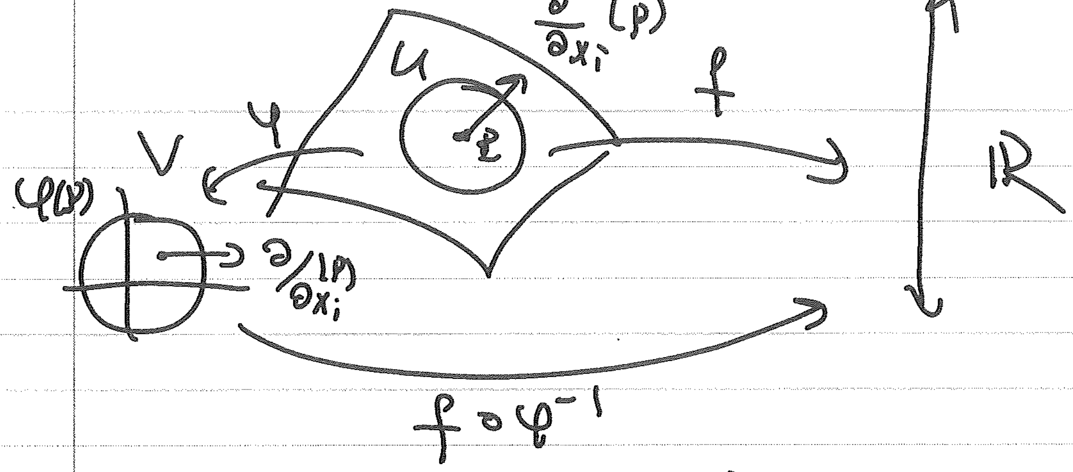
M smooth manifold, $p \in M$, $\varphi : U \to V$

$p \in U \subseteq M$ open, $V \subseteq \mathbb{R}^n$ open

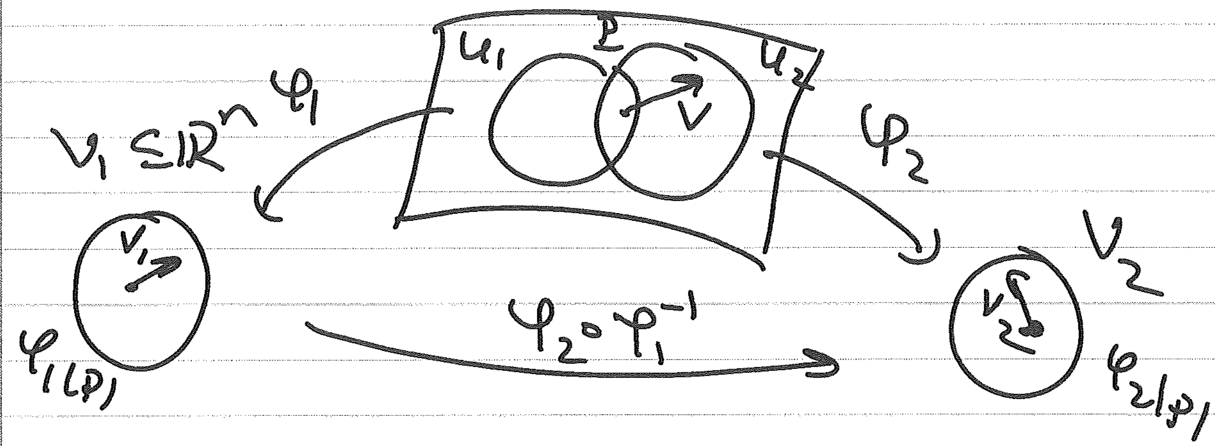
$$T_pM = T_pU \xrightarrow[\cong]{D\varphi_p} T_{\varphi(p)}V = T_{\varphi(p)}\mathbb{R}^n$$

$\varphi = (x_1, x_2, \dots, x_n)$ $\quad T_{\varphi(p)}\mathbb{R}^n = \text{span}\left\{\frac{\partial}{\partial x_i}\Big|_p\right\}_{i=1}^n$

$f : M \to \mathbb{R}$, $\quad \frac{\partial}{\partial x_i}(f)\Big|_p = \frac{\partial}{\partial x_i}(f \circ \varphi^{-1})\Big|_{\varphi(p)}$



$$\frac{\partial}{\partial x_i}(p)(f) \doteq \frac{\partial}{\partial x_i}(f \circ \varphi^{-1})$$



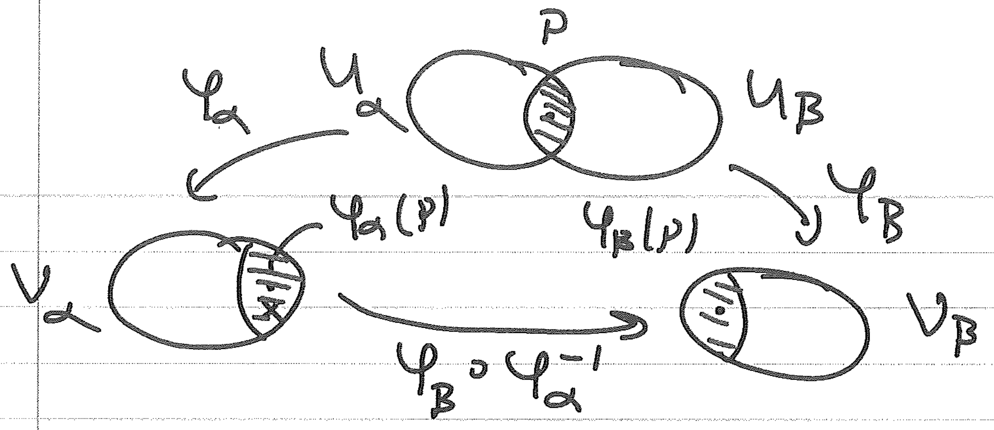
$D\varphi_1(v_1) = v = D\varphi_2(v_2)$ if and only if

$D(\varphi_2 \circ \varphi_1^{-1})_{\varphi_1(p)}(v_1) = v_2$.

<u>Tangent Bundle</u> $M = \bigcup_\alpha U_\alpha$, $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha \in A}$

is an atlas for M.

$$M = \bigcup_\alpha U_\alpha = \dot{\bigcup_\alpha} V_\alpha \Big/ \begin{matrix}(\varphi_\beta \circ \varphi_\alpha^{-1})(x) \sim x \\ \forall x \in \varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta)\end{matrix}$$



$$M = \bigcup_\alpha U_\alpha = \dot{\bigcup_\alpha} V_\alpha \Big/ x \sim (\varphi_\beta \circ \varphi_\alpha^{-1})(x)$$

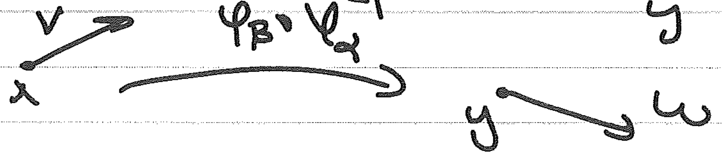
$V_\alpha \times \mathbb{R}^n \subseteq \mathbb{R}^{2n}$ open

$$\underline{T_* M} = \bigcup_\alpha T_* U_\alpha = \dot{\bigcup_\alpha} T_* V_\alpha \Big/ (x, v) \sim (y, w)$$

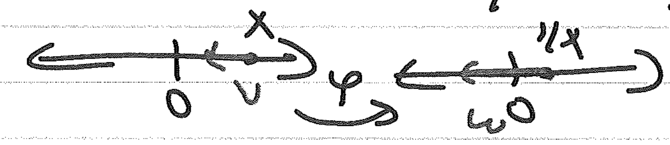
2n-dim'l smooth manifold

$$T_* V_\alpha = V_\alpha \times \mathbb{R}^n \longrightarrow T_* V_\alpha = V_\beta \times \mathbb{R}^n$$

$$(x, v) \longmapsto \Big(\underbrace{(\varphi_\beta \circ \varphi_\alpha^{-1})(x)}_{y}, \underbrace{D(\varphi_\beta \circ \varphi_\alpha^{-1})_{(p)}(v)}_{w}\Big)$$



<u>Example:</u> $S^1 = \mathbb{R} \,\dot{\cup}\, \mathbb{R} \Big/ x \sim \frac{1}{x},\ x \neq 0$



$$T_* S^1 = T_* \mathbb{R} \,\dot{\cup}\, T_* \mathbb{R} \Big/ (x, v) \sim \Big(\frac{1}{x}, \overbrace{\frac{-v}{x^2}}^{w}\Big)$$

$$D\varphi_x(v) = -\frac{1}{x^2}(v) = -\frac{v}{x^2}$$

<u>Vector Fields</u> $T_* M \xrightarrow{\pi} M$ smooth map

$$(x, v) \longmapsto x$$

A vector field on M is a section $s: M \longrightarrow T_x M$. Hence, $\pi \circ s = id_M$

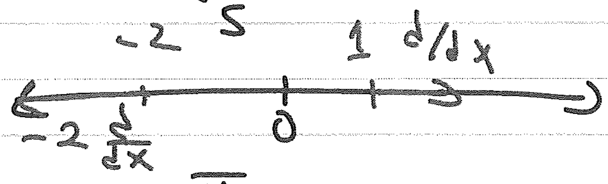
$$T_x M \xrightarrow{\pi} M, \quad s: M \to T_x M$$

$s(x) = (x, v(x))$

$\pi(x, v(x)) = x$

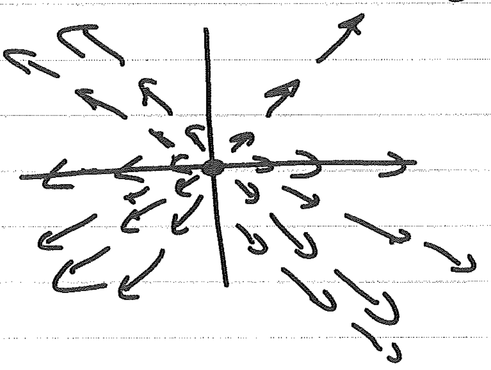
<u>Example</u> $M = \mathbb{R}$, $T_x \mathbb{R} \xrightarrow{\pi} \mathbb{R}$, with $s: \mathbb{R} \to T_x \mathbb{R}$

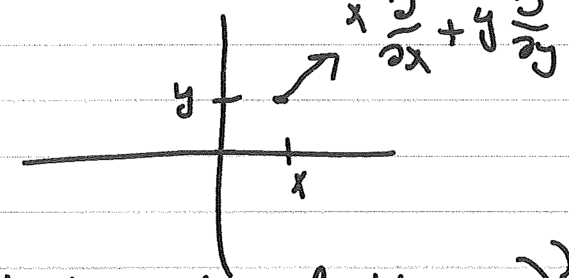
$s(x) = x \dfrac{d}{dx}$



<u>Example</u> $M = \mathbb{R}^2$, $T_x M \xrightarrow{\pi} M$, with $s_1: M \to T_x M$

$s_1(x,y) = x \dfrac{\partial}{\partial x} + y \dfrac{\partial}{\partial y}$

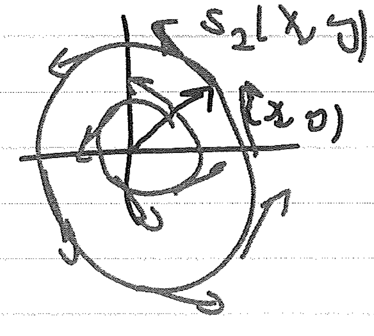




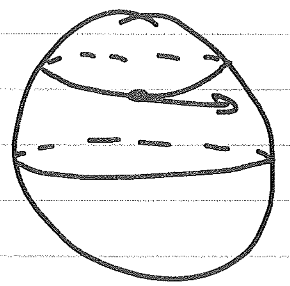
Radial vector field on $\mathbb{R}^2$

$s_2 : M = \mathbb{R}^2 \longrightarrow T_x M = T_x \mathbb{R}^2$

$s_2(x,y) = -y \dfrac{\partial}{\partial x} + x \dfrac{\partial}{\partial y}$



Ex $M = S^2$, $T_x M = T_x S^2$



$(x,y,z) \rightsquigarrow (-y, x, 0)$

$\downarrow$

$-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$

$$s(x,y,z) = \left(-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}\right)\Big|_{(x,y,z)}$$

Math 709, 4-5

Note Title 11.02.2020

$$S^2 = \mathbb{C} \dot\cup \mathbb{C} \big/ z \sim \phi(z) = \frac{1}{z}, \quad z \neq 0 \qquad -\frac{1}{t^2}$$

$$T_* S^2 = T_*\mathbb{C} \dot\cup T_*\mathbb{C} \big/ (z,w) \sim (\phi(z), D\phi(z)(w))$$

$$(z,w) \in T_*\mathbb{C}, \quad z \neq 0$$

$$= \mathbb{C}\times\mathbb{C} \dot\cup \mathbb{C}\times\mathbb{C} \big/ (z,w) \sim \left(\frac{1}{z}, -\frac{1}{z^2}\cdot w\right)$$

$$z \neq 0$$

Ex $S^4 \cong \mathbb{H}P^1 = \mathbb{H} \cup \mathbb{H} / p \sim \frac{1}{p}, \; p \neq 0$

$\mathbb{H} = \mathbb{R}^4, \; p = (x_1, x_2, x_3, x_4) = x_1 + i x_2 + j x_3 + k x_4$

$\bar{p} = x_1 - i x_2 - j x_3 - k x_4$

$p\bar{p} = x_1^2 + x_2^2 + x_3^2 + x_4^2 \Rightarrow p \frac{\bar{p}}{\|p\|} = 1, \; p \neq 0.$

$p^{-1} = \frac{1}{p} = \frac{\bar{p}}{\|p\|}.$

$$T_* S^4 = T_* \mathbb{H} \,\dot{\cup}\, T_* \mathbb{H} \big/ (p, v) \sim \left(\frac{1}{p}, D\phi(p)(v)\right)$$

$$\phi : \mathbb{H}^* \longrightarrow \mathbb{H}^*, \quad \phi(p) = \frac{1}{p}, \quad p \in \mathbb{H}^*$$

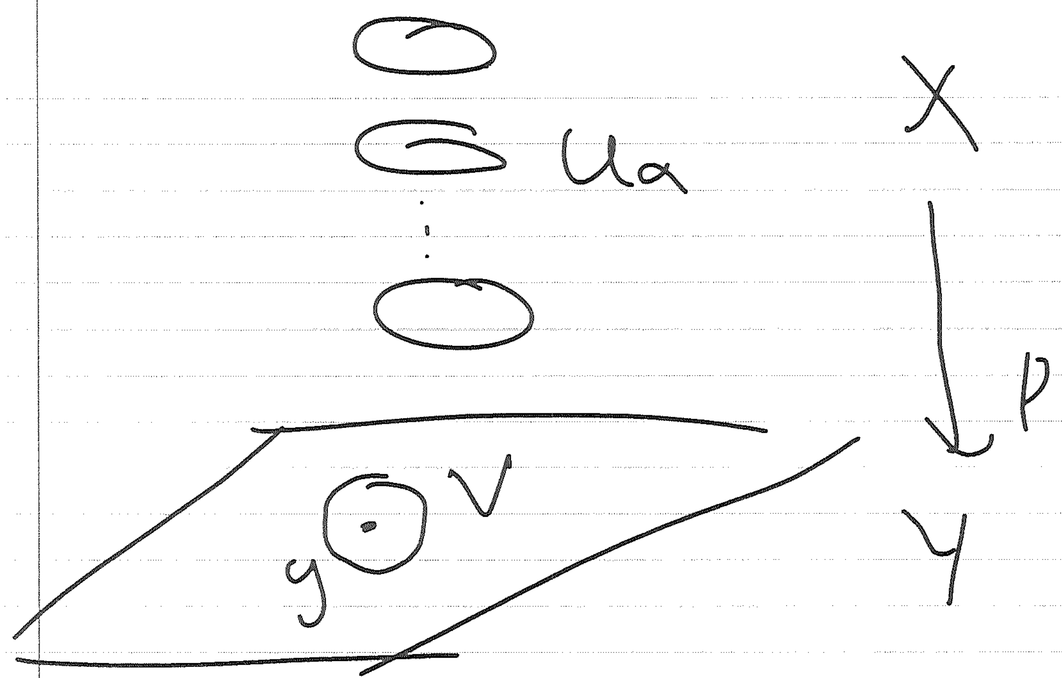
$$D\phi(p)(v) = -\frac{1}{p}\, v\, \frac{1}{p}$$

Qutient Manifolds:

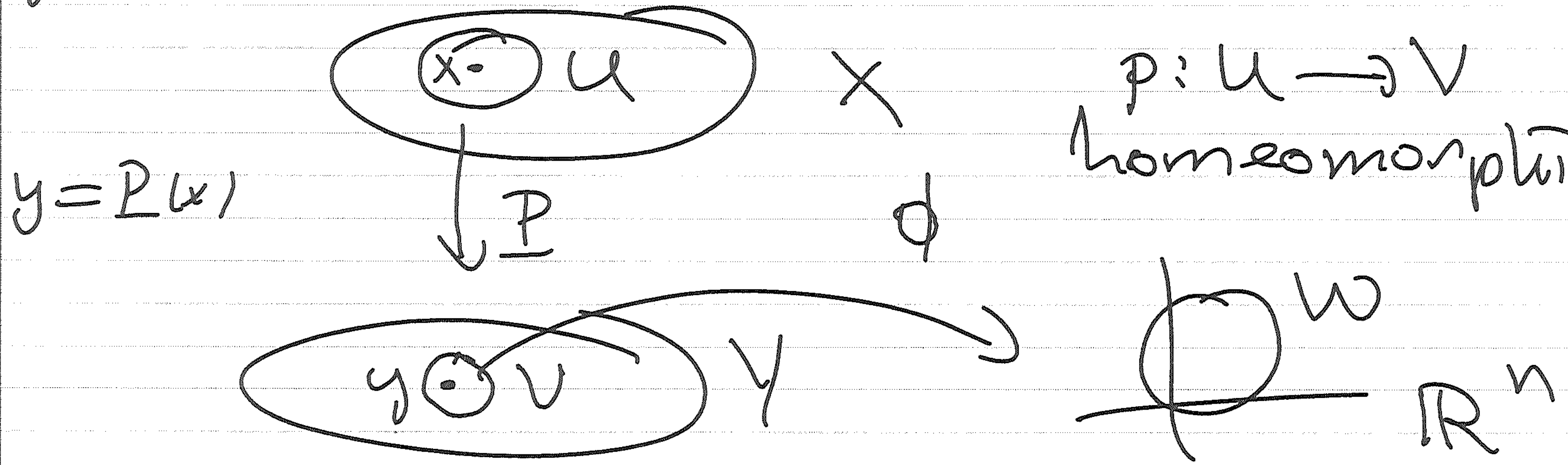
X, Y topological spaces, $p: X \to Y$ cont. map. p is called a covering space if for any $y \in Y$ there is a neighborhood V of Y with

i) $y \in V$

ii) $p^{-1}(V)$ is a disjoint union open subsets $\{U_\alpha\}$ of X, when for each α the restriction map $p: U_\alpha \to V$ is a homeomorphism.

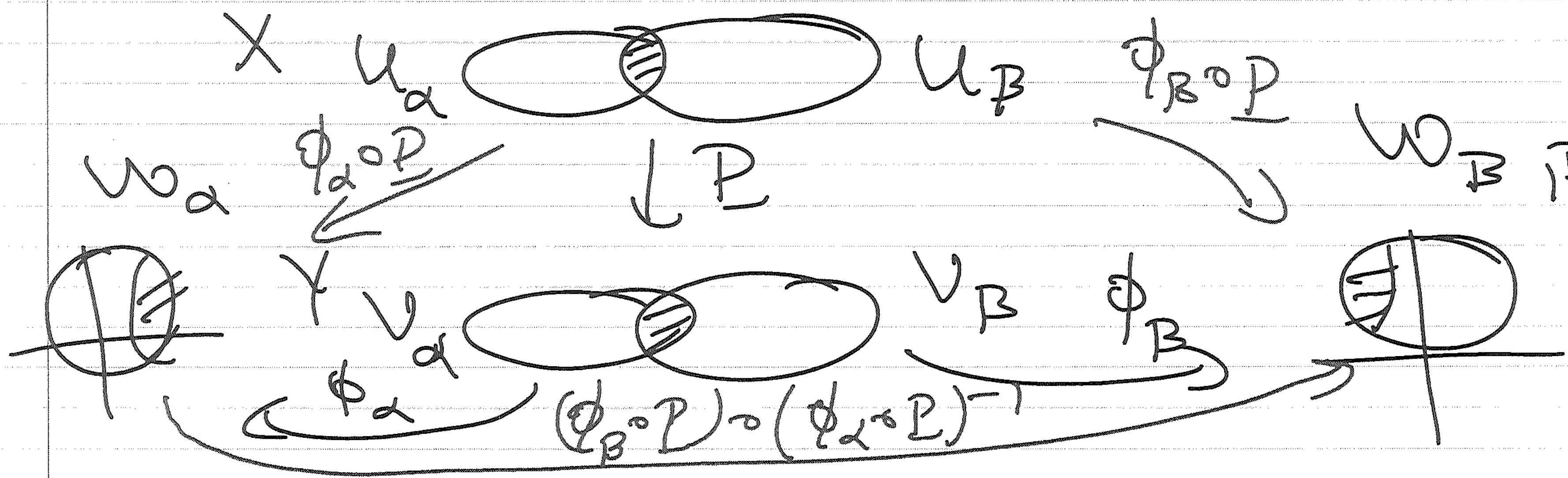

U_a
X
p
Y
g
V

If Y has a smooth structure then X gets a smooth structure as follows:



$p: U \to V$ homeomorphism

The $\phi \circ \underline{P} : U \longrightarrow W$ is a coordinate system about $x \in X$.



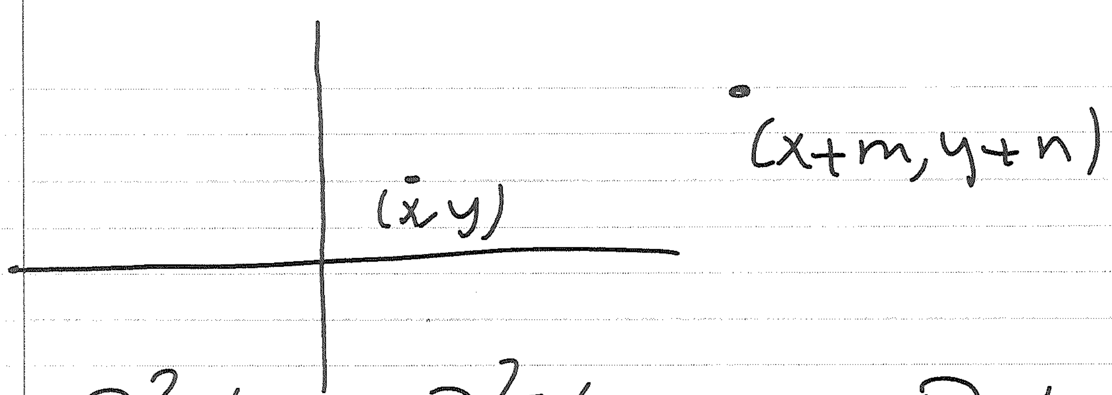
$$(\phi_\beta \circ P) \circ (\phi_\alpha \circ P)^{-1} = \phi_\beta \circ \underbrace{(P \circ P^{-1})}_{= Id} \circ \phi_\alpha^{-1}$$

$$= \phi_\beta \circ \phi_\alpha^{-1} \in C^\infty, \text{ because}$$

Y has smooth structure.

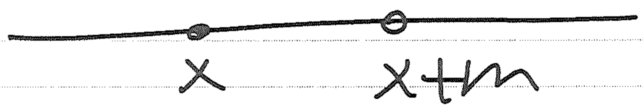
<u>Ex</u>: $G = \mathbb{Z} \times \mathbb{Z}$, $G \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

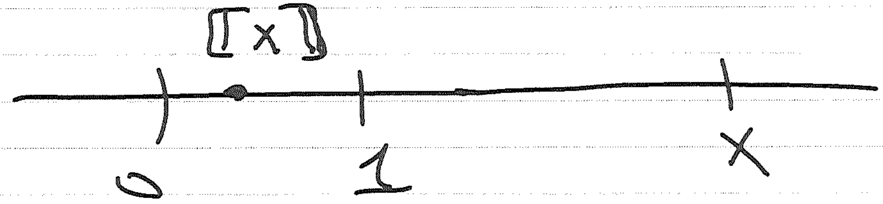
$$(m,n) \cdot (x,y) = (x+m, y+n)$$



$(m,n) \in G = \mathbb{Z} \times \mathbb{Z}$

$$\mathbb{R}^2/G = \mathbb{R}^2/\mathbb{Z}\times\mathbb{Z} = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} = S^1 \times S^1$$





$$\mathbb{R}/\mathbb{Z} \simeq [0,1]/0\sim 1 = S^1$$

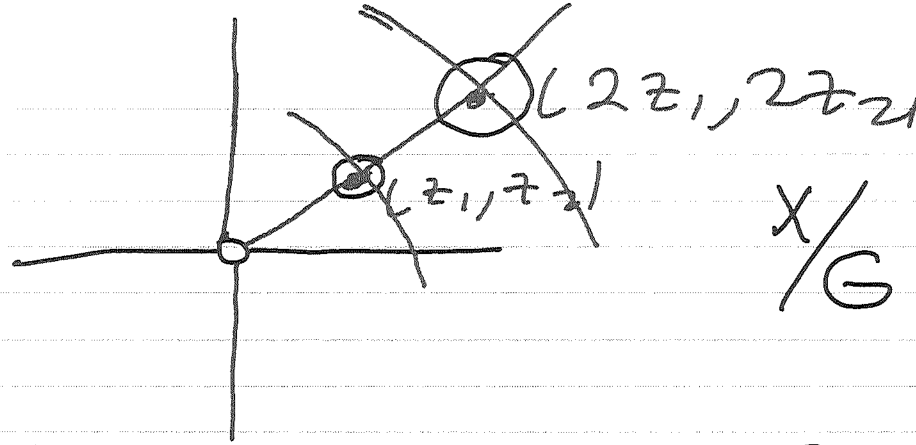
$h : \mathbb{R} \longrightarrow S^1$, $h(t) = (\cos 2\pi t, \sin 2\pi t)$

$h(t) = h(t+m)$ and

$h(t_1) = h(t_2)$ implies $t_1 - t_2 \in \mathbb{Z}$.

2) $G = \mathbb{Z}$, $X = \mathbb{C}^2 \setminus \{(0,0)\}$

$G \times X \longrightarrow X$, $n \cdot (z_1, z_2) = (2^n z_1, 2^n z_2)$



$$X/G \simeq S^3 \times S^1$$

$$\mathbb{C}^2 \setminus \{(0,0)\} / \mathbb{Z} \longrightarrow S^3 \times S^1$$

$$(z_1, z_2) \longmapsto \left(\frac{(z_1, z_2)}{\|(z_1, z_2)\|}, e^{2\pi i \log_2 \|(z_1, z_2)\|} \right)$$

diffeomorphism.

Rank Theorems:

<u>Definition</u>: Let $f: M \to N$ be a smooth map of smooth manifolds and $p \in M$. If $Df(p): T_pM \to T_{f(p)}N$ is injective then we say that f is an <u>immersion</u> at p. If $Df(p): T_pM \to T_{f(p)}N$ is onto then we say that f is an submersion at p.

Ex: $m \leq n$, $f: \mathbb{R}^m \to \mathbb{R}^n$,

$$(x_1, \ldots, x_m) \longmapsto (x_1, \ldots, x_m, \overbrace{0, \ldots, 0}^{n-m})$$

$Df(p): T_p\mathbb{R}^m \to \mathbb{R}^n$, $v = (v_1, \ldots, v_m) \in T_p\mathbb{R}^m$

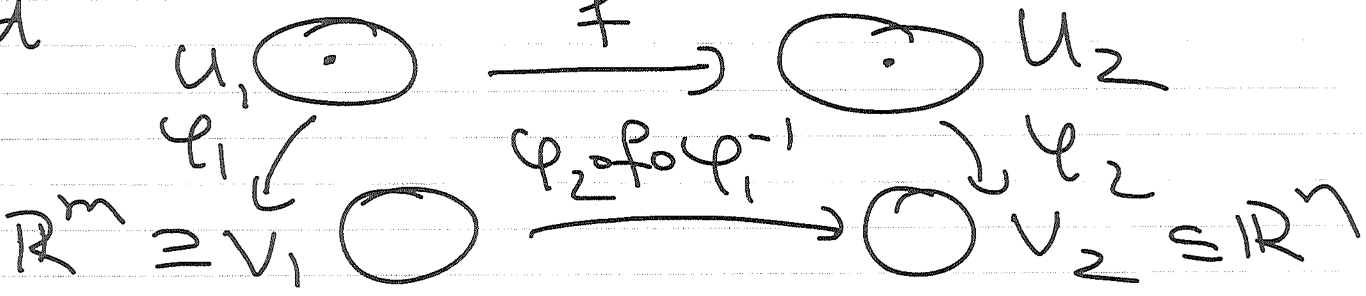
$Df(p)(v) = (v_1, \ldots, v_m, 0, \ldots, 0)$ is clearly 1-1.

This is called the canonical immersion.

If $m \geq n$, $g : \mathbb{R}^m \to \mathbb{R}^n$, $g(x_1, \dots, x_n, \dots, x_m) = (x_1, \dots, x_n)$.

Clearly, $Dg(p) : T_p\mathbb{R}^m \to T_{g(p)}\mathbb{R}^n$ is given by $Dg(p)(v_1, \dots, v_n, \dots, v_m) = (v_1, \dots, v_n)$. So $Dg(p)$ is clearly onto and thus g is an submersion, called the canonical submersion.

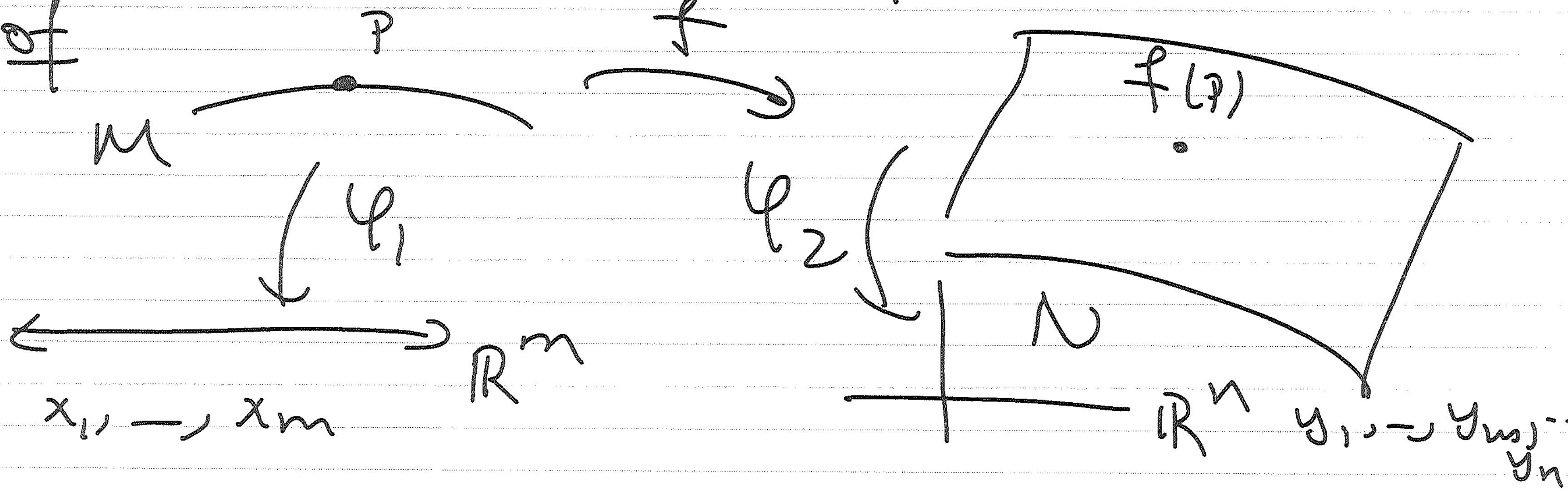
Theorem: Let $f: M \to N$ be a smooth map and $p \in M$ so that f is an immersion at p. Then one can find coordinate charts around p and $f(p)$, say $\varphi_1 : U_1 \to V_1$, $\varphi_2 : U_2 \to V_2$, $p_1 \in U_1 \subseteq M$, $f(p) \in U_2 \subseteq N$, $V_1 \subseteq \mathbb{R}^m$, $V_2 \subseteq \mathbb{R}^n$, so that



$(\varphi_2 \circ f \circ \varphi_1^{-1})(x_1, \dots, x_m) = (x_1, \dots, x_n, 0, \dots, 0).$

Similar statements holds for submersions.

Proof



$Df(p) : T_pM \longrightarrow T_{f(p)}N$ is injective

$$Df(p) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_m} \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{bmatrix}_{n \times m}$$

$f = (f_1, \ldots, f_n)$

$f_i = f_i(x_1, \ldots, x_m)$

Assume that the first m-rows of $Df(p)$ are linearly independent.

let $g : \mathbb{R}^m \times \mathbb{R}^{n-m} \longrightarrow \mathbb{R}^n$ be given by

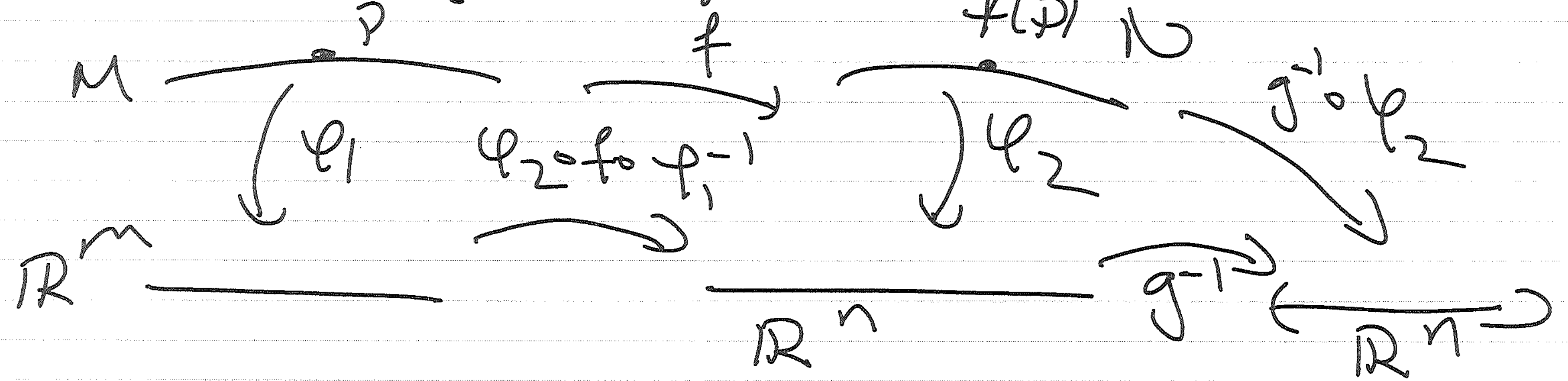
$$g(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n) \doteq f(x_1, \ldots, x_m) + (0, \ldots, 0, x_{m+1}, \ldots, x_n)$$

$$= (f_1, f_2, \ldots, f_m, f_{m+1} + x_{m+1}, \ldots, f_n + x_n)$$

$$Dg(p) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_m} & 0 & 0 & \cdots & 0 \\ * & & * & 1 & 0 & \cdots & 0 \\ * & \cdots & * & 0 & 1 & \cdots & 0 \\ * & & * & & & & 1 \end{bmatrix}$$

Now Dg is invertible, $g : \underbrace{\mathbb{R}^m \times \mathbb{R}^{n-m}}_{\mathbb{R}^n} \to \mathbb{R}^n$.

So by the Inverse function theorem g is a local diffeomorphism.



$(x_1, \ldots, x_m) \longmapsto (f_1, \ldots, f_n) \longmapsto g^{-1}(f_1, \ldots, f_n)$, where

$$(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n) \underset{g^{-1}}{\overset{g}{\rightleftarrows}} (f_1, \ldots, f_m, f_{m+1} + x_{m+1}, \ldots, f_n + x_n)$$

<u>Claim:</u> $g^{-1}(f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m)) = (x_1, \ldots, x_m, 0, \ldots, 0)$

Because, $g(x_1, \ldots, x_m, 0, \ldots, 0) = (f_1, \ldots, f_m, f_{m+1}, \ldots, f_n)$.

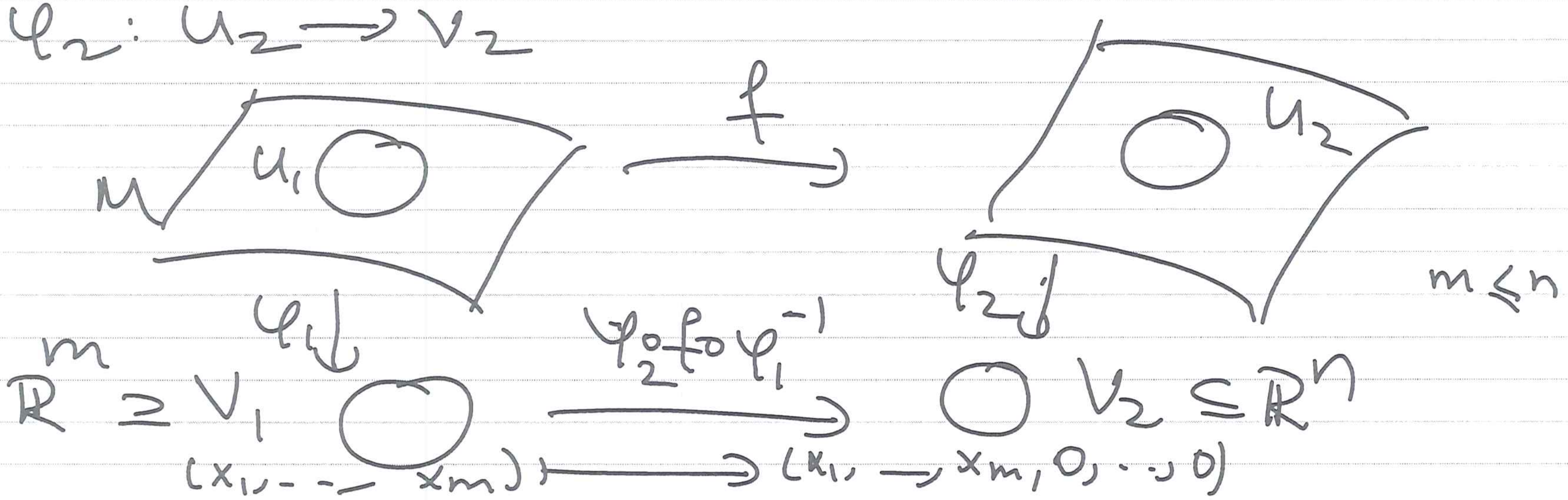
math709, 6, 7

Note Title 12.02.2020

$f: M \to N$ immersion at a point $p \in M$

$p \in U_1 \subseteq M$, $f(p) \in U_2 \subseteq N$, $\varphi_1: U_1 \to V_1$

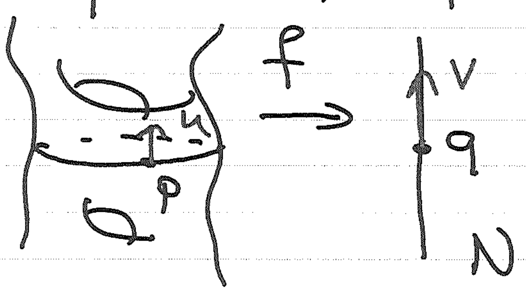
$\varphi_2: U_2 \to V_2$



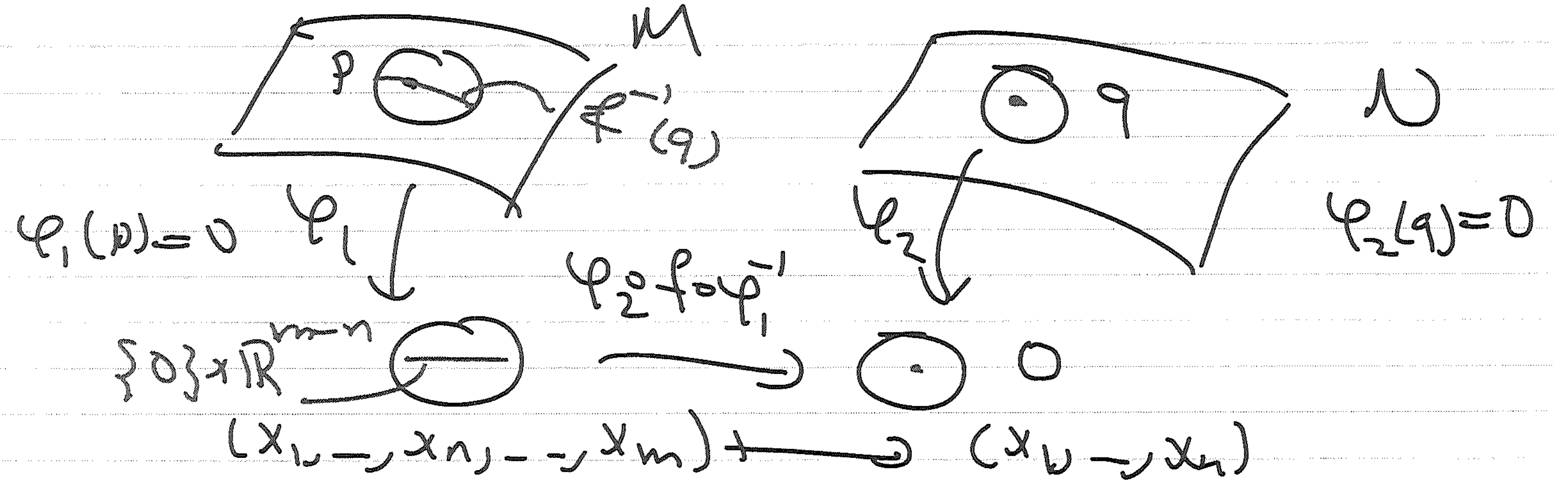
If f is a submersion then φ_i's can be chosen that $\varphi_2 \circ f \circ \varphi_1^{-1}$ becomes standard submersion.

$$\varphi_2 \circ f \circ \varphi_1^{-1}(x_1, \dots, x_n, \dots x_m) = (x_1, \dots, x_n) \qquad (m \geq n).$$

<u>Definition:</u> $f : M \to N$ smooth map. A point $q \in N$ is called a regular value if $Df(p) : T_pM \to T_qN$ is onto for all $p \in f^{-1}(q)$.



In this case, since $Df(p): T_pM \to T_qN$ is onto, f is an submersion at $p \in M$ and in some coordinate systems



So, $f^{-1}(q)$ becomes in this coordinate system

$$\varphi_1(f^{-1}(q)) = \{(x_1, \dots, x_n, \dots x_m) \in \mathbb{R}^m \mid x_1 = \cdots = x_n = 0\}$$
$$= \{0\} \times \mathbb{R}^{m-n}$$

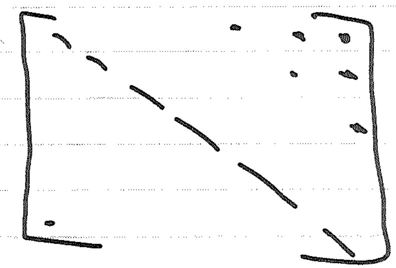
This implies that $f^{-1}(q)$ is an $m-n$-dimensional submanifold of M.

<u>Example:</u> $\tilde{\Phi} : M(n,n) \longrightarrow S(n)$, $\tilde{\Phi}(Q) = Q^T Q$

$M(n,n) = \mathbb{R}^{n \times n} = \mathbb{R}^{n^2}$: the set of all $n \times n$-real matrices

$S(n) = \{A \in M(n,n) \mid A^T = A\}$, the set of all

$n \times n$ - symmetric real matrices.



$$S(n) = \left\{ [a_{ij}]_{n\times n} \;\middle|\; a_{ij} = a_{ji} \;\; \forall i,j \right\}$$

$$= \mathbb{R}^{\frac{n(n+1)}{2}}$$

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\Phi : M(n,n) \longrightarrow S(n), \quad Q \longmapsto Q^T Q.$$

$$D\Phi(I_d) : T_{I_d} M(n,n) \longrightarrow T_{I_d} S(n)$$

$$\qquad\qquad\quad \| \qquad\qquad\qquad\qquad \|$$

$$\qquad\quad M(n,n) \qquad\qquad\quad S(n)$$

$$A \longmapsto A + A^T$$

<u>$D\Phi$ is onto:</u> If $Q \in S(n)$, then $Q = \frac{Q}{2} + \frac{Q^T}{2} = D\Phi_{(Id)}\left(\frac{Q}{2}\right)$.

Hence, $Id \in S(n)$ is a regular value of Φ.

Therefore, $\Phi^{-1}(Id)$ is a smooth submanifold of $M(n,n)$ of dimension $\dim M(n,n) - \dim S(n) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

Moreover,

$$\Phi^{-1}(Id) = \{Q \in M(n,n) \mid Q^T Q = Id\}$$

$= O(n)$ the set of $n \times n$ orthogonal real matrices.

Sard's Theorem: $\Omega \subseteq \mathbb{R}^n$ open subset, $F: \Omega \to \mathbb{R}^m$ smooth map. Let $C \subseteq \Omega$ be the set of all critical points of F: $C = \{p \in \Omega \mid DF(p): T_p\Omega \to T_{F(p)}\mathbb{R}^m$ is not onto$\}$.
Then the $F(C)$ has measure zero in $\mathbb{R}^m$.

Ex: $n < m$, $DF(p): T_p\Omega \longrightarrow T_{F(p)}\mathbb{R}^m$ cannot be onto
($T_p\Omega \cong \mathbb{R}^n$, $T_{F(p)}\mathbb{R}^m \cong \mathbb{R}^m$)

Hence, the set of critical points C of F in Ω is Ω. So by Sard's Theorem $F(C)$, the set of critical values of F has measure zero. So the image $F(\Omega)$ has measure zero.

An application of Sard's Theorem: Embedding of smooth manifolds into Euclidean spaces.

<u>Theorem:</u> Let M be a smooth n-dimensional manifold. Then there is an immersion of M into $\mathbb{R}^{2n}$ and an embedding into $\mathbb{R}^{2n+1}$.

<u>Idea:</u> First embedd M into $\mathbb{R}^N$ for some big N.

M compact $p \in M$, $p \in U \subseteq M$, $\varphi : U \to V \subseteq \mathbb{R}^n$

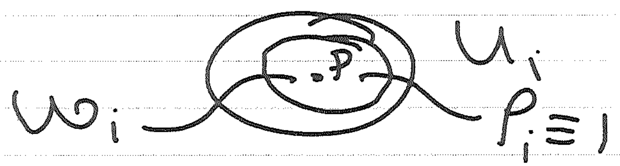
$M = U_1 \cup \cdots \cup U_k$, $\varphi_i : U_i \longrightarrow V_i \subseteq \mathbb{R}^n$

Extend φ_i to all M as a smooth function.

Then $M \longrightarrow \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n = \mathbb{R}^{nk}$

$p \longmapsto (\varphi_1(p), \varphi_2(p), \ldots, \varphi_k(p))$

To make this map also one to one we choose for each i a smooth function $\rho_i : M \longrightarrow [0,1]$ so that it takes the value 1 in an open set in U_i containing p.



$W_i \subset \overline{W_i} \subset U_i$

$M = \cup W_i$

Then the map $F : M \longrightarrow \mathbb{R}^N$,

$F = (\varphi_1, \ldots, \varphi_k, f_1, f_2, \ldots, f_k)$ is a 1-1 immersion.

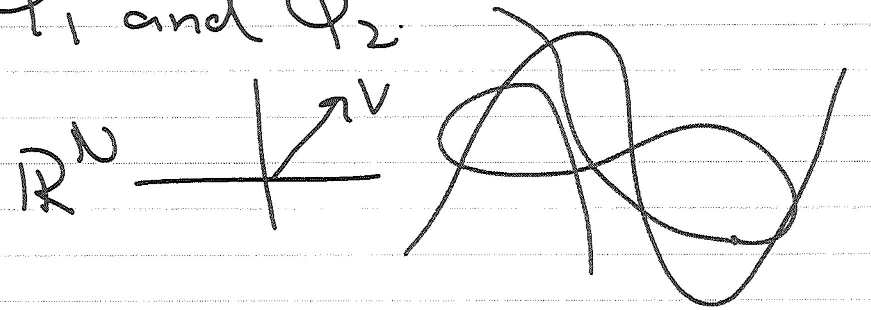
Moreover, M is compact and thus any 1-1 immersion is an embedding.

So we may assume M is a submanifold of some $\mathbb{R}^N$.
Assume that $N > 2n+1$. Consider the functions

$\varphi_1 : M \times M \times \mathbb{R} \longrightarrow \mathbb{R}^N$, $(x_1, x_2, \lambda) \longmapsto \lambda(x_1 - x_2)$ and

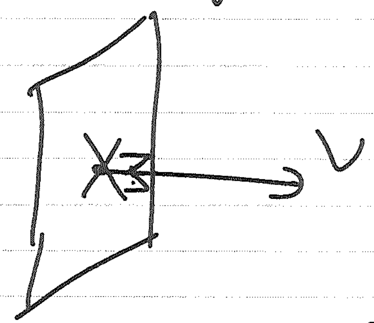
$\varphi_2 : T_x M \longrightarrow \mathbb{R}^N$, $(x, v) \longmapsto v$, $(x, v) \in T_x M \subseteq T_x \mathbb{R}^N \subseteq \mathbb{R}^N$

$N > 2n+1 > 2n$ and thus $\operatorname{Im}\psi_1$ and $\operatorname{Im}\psi_2$ consists of critical values of ψ_1 and ψ_2, respectively.
In particular, ψ_1 and ψ_2 have measure zero.
Since any ball in $\mathbb{R}^N$ has positive measure there is some vector $v \in \mathbb{R}^N$, $v \neq 0$, not contained in the images of ψ_1 and ψ_2.



Let Γ_v be the hypersurface in $\mathbb{R}^n$ perpendicular to the vector v.

$\mathbb{R}^{n-1} \simeq \Gamma_v$



Let $\pi : \mathbb{R}^n \longrightarrow \Gamma_v \simeq \mathbb{R}^{n-1}$ be the orthogonal projection.

<u>Claim:</u> The composition

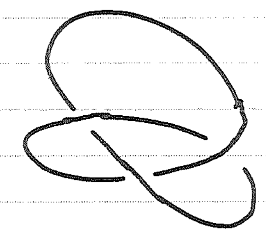
$$\pi \circ F : M \longrightarrow \Gamma_v \simeq \mathbb{R}^{n-1}$$

is still a one-to-one immersion.

Proof Exercise.

This finishes the proof. ∎

Ex



Ex $S^1 \times S^3 = \mathbb{C}^2 \setminus \{(0,0)\} / (z_1, z_2) \sim (2z_1, 2z_2)$

$\cap$

$\mathbb{R}^2 \times \mathbb{R}^4 = \mathbb{R}^6$

$S^1 \times S^3 \not\subset \mathbb{C}^N$ as a complex submanifold. ($S^1 \times S^3$ compact)

$S^1 \times S^3 \not\subset \mathbb{CP}^N$ as a complex submanifold.
(Homology of $S^1 \times S^3$ is <u>not</u> suitable)

<u>Differential Forms:</u> $U \subseteq \mathbb{R}^n$ open subset

$p \in U$, T_pU tangent space: $T_pU = \text{span}\left\{ \frac{\partial}{\partial x_1}\Big|_p, \dots, \frac{\partial}{\partial x_n}\Big|_p \right\}$

Cotangent space $T_p^*U = \text{dual of } T_pU$

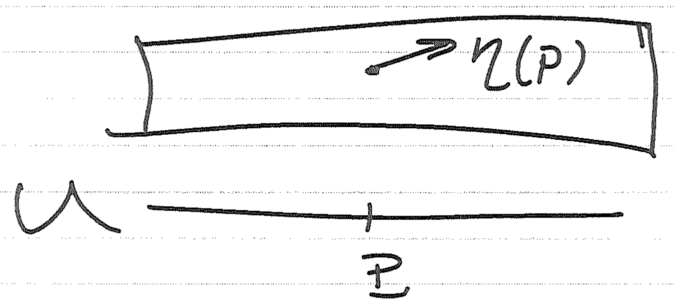
$= \text{span}\{ dx_1|_p, \dots, dx_n|_p \}$ such that

$$dx_i(p)\left(\frac{\partial}{\partial x_j}\Big|_p\right) = \delta_{ij}, \text{ for all } i,j = 1 \to n.$$

$$\begin{array}{ccc} T^*U = \bigcup_{p \in U} T_p^* U & \ni & (p, \eta) \\ \pi \downarrow & & \downarrow \\ U & & p \end{array}$$

As in the case of tangent bundle the cotangent bundle is a smooth manifold of dimension $2n$ and the projection map π is smooth.

A 1-form on U is a smooth section of the map $\pi: T^*U \to U$: $\eta: U \to T^*U$ such that $\pi \circ \eta: U \to U$ is identity.



$\eta(p) = a_1(p)\, dx_1|_p + \cdots + a_n(p)\, dx_n|_p$

$a_i: U \to \mathbb{R}$ smooth functions

<u>Ex</u>: $M = \mathbb{R}^3$, $\omega(x,y,z) = x^2 y\, \underline{dx} - 3e^x z\, \underline{dy} + dz$ is a smooth one form on $\mathbb{R}^3$.

If $X(x,y,z) = x \frac{\partial}{\partial x}\Big|_p - 2\frac{\partial}{\partial y}\Big|_p - y\frac{\partial}{\partial z}\Big|_p$, then the composition $\omega(X)(x,y,z)$ is a function on $\mathbb{R}^3$ given by

$$\omega(X)(x,y,z) = x^3 y + 6e^x z - y$$

k-form on $U \subseteq \mathbb{R}^n$ $\qquad \underline{dx_i} = \underline{dx_i}|_p$

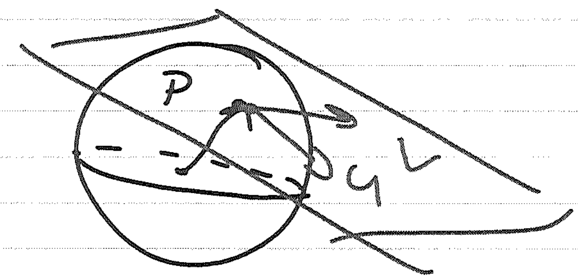
$f(p)\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$

In general, a k-form on U has the form

$$\omega = \sum_I f_I(p)\, dx_I, \quad I = (i_1 < i_2 < \cdots < i_k) \quad 1 \leq i_j \leq n$$

$dx_I \doteq dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$.

<u>**Örnek:**</u> S^2 unit sphere in $\mathbb{R}^3$. Let ω be the 2-form on S^2 given by



$$\omega(p)(u,v) \doteq (u \times v) \cdot p$$

$u, v \in T_p S^2$

$p = (x, y, z)$, $\quad u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$

$$\omega(p)(u,v) = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1) \cdot (x, y, z)$$

$$\Rightarrow \omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$$

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$U \subseteq \mathbb{R}^n$ ω k-form on U

$\omega = \sum f_I \, dx_I$ $I = (i_1, \ldots, i_k)$ $1 \leq i_1 < i_2 < \cdots < i_k \leq n$

$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$

$f_I : U \to \mathbb{R}$ smooth function

<u>Wedge product</u>

$(f_I \, dx_I) \wedge (g_J \, dx_J) = f_I g_J \, dx_I \wedge dx_J$

and for general forms we extend this definition linearly.

If ω is a k-form and ν is an l-form then $\omega \wedge \nu = (-1)^{kl} \nu \wedge \omega$.

$(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) \wedge (dx_{j_1} \wedge \cdots \wedge dx_{j_l})$

$= \underbrace{(-1)^k (-1)^k \cdots (-1)^k}_{l} \, dx_{j_1} \wedge \cdots \wedge dx_{j_l} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$

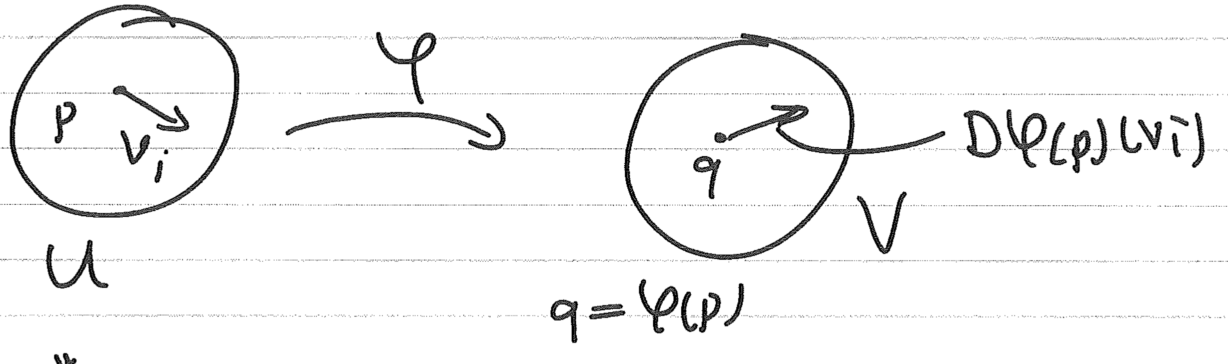
<u>Exterior Derivation</u>

$f_I \, dx_I$ k-form, $d(f_I \, dx_I) \doteq df_I \wedge dx_I$

Fact: $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$,
where ω is a k-form and η is a l-form.

Pull back: $\varphi : U \to V$, $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$
φ smooth function. Let ω be a k-form on V. Then $\varphi^*(\omega)$ is a k-form on U defined by

$$\varphi^*(\omega)(v_1, \dots, v_k) \doteq \omega(D\varphi(p)(v_1), \dots, D\varphi(p)(v_k))$$



φ^* commutes with d and wedge product.

- $\varphi^*(d\omega) = d(\varphi^*(\omega))$
- $\varphi^*(\omega \wedge \eta) = \varphi^*(\omega) \wedge \varphi^*(\eta)$.

Practically we compute φ^* as follows:

$\varphi : U \to V$ $\quad \mathbb{R}^n \; x_1, \ldots, x_n$

$U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ $\quad \mathbb{R}^m \; y_1, \ldots, y_m$

$\omega \in \Omega^k(V)$, $\omega = \sum f_I \, dy_I$

$\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_m) = (y_1, \ldots, y_m)$

$y_i = \varphi_i(x_1, \ldots, x_n)$, $dy_i = d\varphi_i$

Hence, $dy_I = d\varphi_I = d\varphi_{i_1} \wedge \cdots \wedge d\varphi_{i_k} \in \Omega^k(U)$.

Example: $\varphi : \underset{u,v}{\mathbb{R}^2} \to \underset{x,y,z}{\mathbb{R}^3}$, $\varphi(u,v) = (\underset{x}{u+v}, \underset{y}{u \cdot v}, \underset{z}{v^2+u})$

$\omega \in \Omega^2(\mathbb{R}^3)$, $\omega = x \, dx \wedge dy - (z+y) \, dy \wedge dz$
$+ 3 \, dz \wedge dx$.

$\varphi^*(\omega)$?

$x = x(u,v) = u+v$, $dx = du + dv$
$y = y(u,v) = u \cdot v$, $dy = v \, du + u \, dv$

$z = z(u,v) = v^2 + u$, $dz = du + 2v \, dv$

So, $\varphi^*(\omega) = x \, (du + dv) \wedge (v \, du + u \, dv)$

$- (z+y) (v \, du + u \, dv) \wedge (du + 2v \, dv)$
$+ 3 \, (du + 2v \, dv) \wedge (du + dv)$

$= xu \, du \wedge dv - xv \, du \wedge dv$

$- (z+y) (2v^2 \, du \wedge dv - u \, du \wedge dv)$
$+ 3 \, (du \wedge dv - 2v \, du \wedge dv)$

$= (xu - xv - (z+y)2v^2 + (z+y)u + 3 - 2v)\, du \wedge dv$

$= \big[(u+v)u - (u+v)v - (v^2u+uv)2v^2$

$\quad + (v^2u+uv)u + 3 - 2v\big]\, du \wedge dv.$

<u>Example:</u> $S^1 = \{(x,y,0) \mid x^2+y^2=1\} \subseteq \mathbb{R}^3$

$\omega = \dfrac{x\,dy - y\,dx}{x^2+y^2} \in \Omega^1(\mathbb{R}^2 \setminus \{(0,0)\})$.

$\varphi : \mathbb{R}^3 \setminus S^1 \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$ $(x,y,z) \longmapsto (x^2+y^2-1, z)$.

$u = x^2+y^2-1$, $v = z$, $\omega = \dfrac{u\,dv - v\,du}{u^2+v^2}$

$du = 2x\,dx + 2y\,dy$, $dv = dz$.

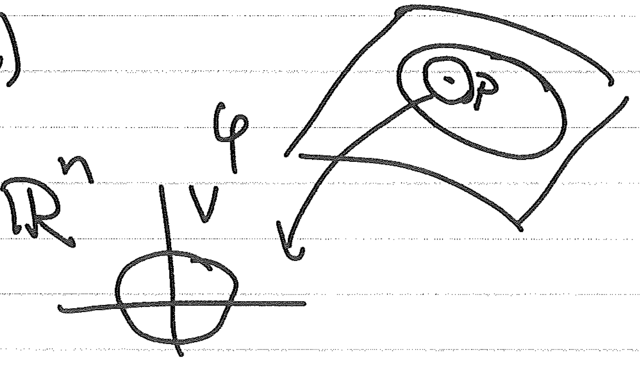
$$\varphi^*(\omega) = \frac{(x^2+y^2-1)\,dz - z(2x\,dx + 2y\,dy)}{(x^2+y^2-1)^2 + z^2}$$

$\varphi^*(\omega) \in \Omega^1(\mathbb{R}^3 \setminus S^1)$.

<u>Forms on Manifolds</u>

M^n smooth manifold, $U \subseteq M$ open subset

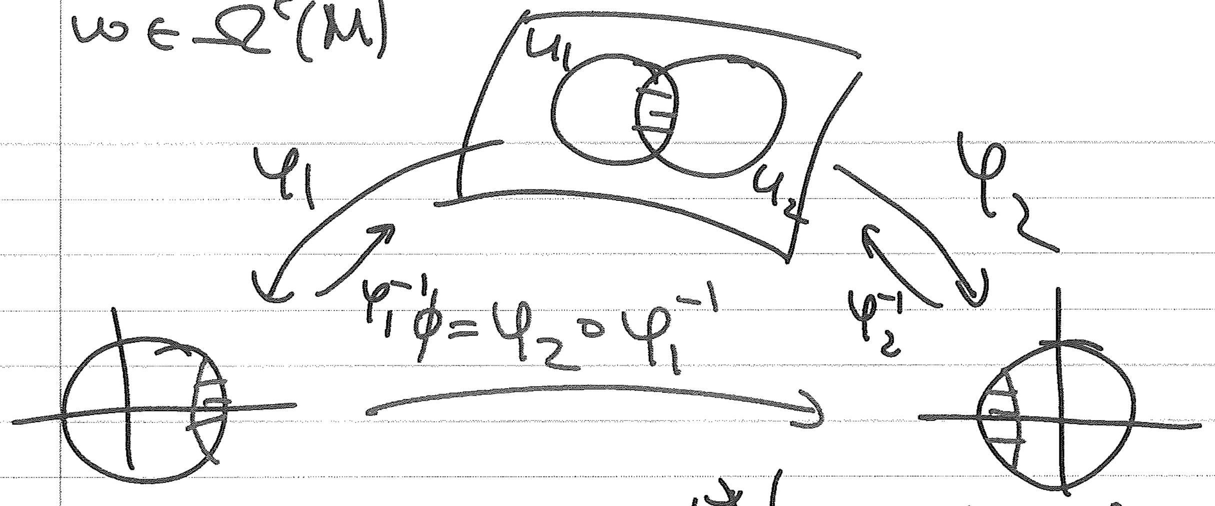
$\omega \in \Omega^k(U)$



$\omega = \varphi^*(\eta)$

$\eta \in \Omega^k(V)$.

$\omega \in \Omega^k(M)$



$$(\varphi_1^{-1})^*(\omega) = \phi^*\left((\varphi_2^{-1})^*(\omega)\right)$$

<u>Example:</u> $\phi: \mathbb{R}^2 \to \mathbb{R}^4$, $(t_1, t_2) \mapsto (x_1, y_1, x_2, y_2)$

$$\phi(t_1,t_2) = (\underbrace{\cos t_1}_{x_1}, \underbrace{\sin t_1}_{y_1}, \underbrace{\cos t_2}_{x_2}, \underbrace{\sin t_2}_{y_2})$$

$\phi(\mathbb{R}^2) = S^1 \times S^1$ $\quad \omega_i \in \Omega^1(\mathbb{R}^4)$

$$\omega_1 = \frac{x_1 dy_1 - y_1 dx_1}{x_1^2 + y_1^2}, \quad \omega_2 = \frac{x_2 dy_2 - y_2 dx_2}{x_2^2 + y_2^2}$$

$$\varphi^*\omega_1 = (\cos t_1)(\cos t_1)\,dt_1 - (\sin t_1)(-\sin t_1)\,dt_1$$
$$= dt_1$$

Similarly, $\phi^*\omega_2 = dt_2$.

So $\varphi^*(\omega_1 \wedge \omega_2) = dt_1 \wedge dt_2$.

<u>Lemma:</u> $d_k: \Omega^k(M) \to \Omega^{k+1}(M)$, $k \geq 0$

Then $d_{k+1} \circ d_k = 0$.

Orientation of Manifolds

Orientation of Vector spaces: Let β and β' be two ordered bases for a vector space V (finite dim'l.)

$\beta = (v_1, \dots, v_n)$, $\beta' = (u_1, \dots, u_n)$.

We say that β and β' are equivalent if the base change matrix $[I]_{\beta}^{\beta'}$ has positive determinant.

$u_1 = a_{11} v_1 + a_{12} v_2 + \dots + a_{1n} v_n$

$\vdots$

$u_i = a_{i1} v_1 + a_{i2} v_2 + \dots + a_{in} v_n$

$\vdots$

$u_n = a_{n1} v_1 + a_{n2} v_2 + \dots + a_{nn} v_n$

$$[I]_{\beta}^{\beta'} = [a_{ij}]$$

$$[I]_{\beta'}^{\beta} = \left([I]_{\beta}^{\beta'}\right)^{-1}$$

This relation on the set of all ordered bases of V is an equivalence relation:

i) $\beta \sim \beta' \Rightarrow \beta' \sim \beta$ (Symmetric)

ii) $\beta \sim \beta$, $[I]_\beta^\beta = Id$, $\det([I]_\beta^\beta) = 1$

($\sim$ is reflexive)

iii) $\beta \sim \beta'$ and $\beta' \sim \beta''$, then

$$[I]_\beta^{\beta''} = [I]_{\beta'}^{\beta''}[I]_\beta^{\beta'} \Rightarrow \det([I]_\beta^{\beta''}) > 0.$$

($\sim$ is transitive)

<u>Remark</u>: Orientation on a 0-dimensional vector space is just an assignment of $\pm$ sign.

An orientation on a vector space V is a choice of an equivalence class of the equivalence relation defined above.

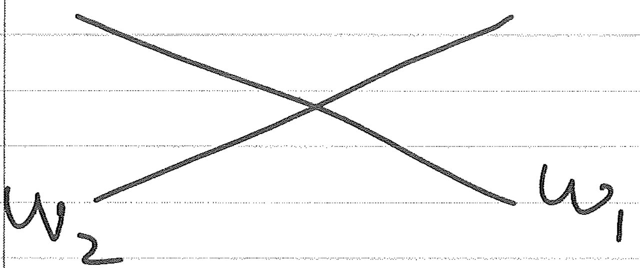
On a vector space there are two orientations.

An orientation of a subspace W of V is just an orientation of W as a vector space.

Let W_1 and W_2 be two <u>oriented</u> subspaces of an oriented space V so that

$W_1 + W_2 = V$ and $\dim V = \dim W_1 + \dim W_2$.

In particular, $W_1 \cap W_2 = \{0\}$.



The orientation of the intersection $\{0\} = W_1 \cap W_2$ is defined as follows:

Let $\beta_1 = (u_1, \ldots, u_k)$ and $\beta_2 = (u_{k+1}, \ldots, u_n)$ be oriented bases for W_1 and W_2.

$W_1 + W_2 = V \Rightarrow \beta = (u_1, \ldots, u_k, u_{k+1}, \ldots, u_n)$ is a basis for V. If the orientation on V given by β is the same as the orientation of V then the sign of the intersection $\{0\}$ is defined to be "+". Otherwise, it is defined to be "−".

__Ex__ $V = \mathbb{R}^4 = (e_1, e_2, e_3, e_4)$

$W_1 = \text{span}(e_1, -e_2)$, $W_2 = (e_3, e_4)$.

$W_1 \cap W_2 = \{0\}$ $(e_1, -e_2, e_3, e_4) \overset{?}{\sim} (e_1, e_2, e_3, e_4)$

No!

Hence, the orientation on $W_1 \cap W_2$ is "$-$".

Remark: Note that the orientation on $W_1 \cap W_2$ may be different than the orientation on $W_2 \cap W_1$.

Example: $V = \mathbb{R}^2 = (e_1, e_2)$

$W_1 = (e_1)$, $W_2 = (e_2)$

$W_1 \cap W_2 \to$ "$+$", but $W_2 \cap W_1 \to$ "$-$"

Now consider the case, where $\dim W_1 + \dim W_2 >$ $\dim V$, and $W_1 + W_2 = V$.

Let $(v_1, v_2, \dots, v_k)$ be an ordered bases for $W_1 \cap W_2$.

$W_1 = (u_1, \dots, u_\ell, v_1, \dots, v_k)$, $\dim W_1 = k + \ell$

$W_2 = (v_1, \dots, v_k, u_{\ell+1}, \dots, u_{n-k})$, $\dim W_2 = n - \ell$

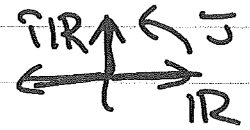
Then $(u_1, \dots, u_\ell, v_1, \dots, v_k, u_{\ell+1}, \dots, u_{n-k})$ is an oriented basis for V. If this basis gives the right orientation on V then the orientation of $W_1 \cap W_2$ is the one given by $(v_1, \dots, v_k)$. Otherwise, the orientation on $W_1 \cap W_2$ is given by $(-v_1, v_2, \dots, v_k)$.

<u>Orientations on Complex Vector spaces</u>

V is an n-dim'l complex vector space. Then V is a $2n$-dimensional real vector space together with an complex structure $J : V \to V$ such that $J^2 = -I$.

<u>Ex</u> $V = \mathbb{C}^n = \mathbb{R}^{2n}$ $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$

$n = 1$, $\mathbb{C} = \mathbb{R}^2 = \{(1,0), (0,1)\}$



$J(1,0) = (0,1) = i(1,0)$.

$J^2(1,0) = i^2(1,0) = (-1,0)$

<u>Fact</u>: A complex vector space, regarded as a real vector space, has a canonical orientation.

<u>Proof</u>: $\dim_{\mathbb{R}} V = 2n$, $J : V \to V$, $J^2 = -Id$.

Let $v_1 \in V$, $v_1 \neq 0$. Then let $v_2 = J v_1$.

If $\dim_{\mathbb{R}} V > 2$ then choose $v_3 \in V \setminus \text{span}\{v_1, v_2\}$

Then let $v_4 = \bar{J} v_3$.

<u>Claim:</u> $\{v_1, v_2, v_3, v_4\}$ is $\mathbb{R}$-linearly independent.

In general if $\dim V = n$, then we can find a basis ($\mathbb{R}$ basis) for V of the form $(v_1, \bar{J}v_1, v_2, \bar{J}v_2, \ldots, v_n, \bar{J}v_n)$.

<u>Claim:</u> If $(u_1, \bar{J}u_1, u_2, \bar{J}u_2, \ldots, u_n, \bar{J}u_n)$ is another such basis then these two bases define the same orientation

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<u>Last time:</u> V complex vector space of dim n.

$\dim_{\mathbb{R}} V = 2n$, $J: V \to V$, $J^2 = -I$.

<u>Claim:</u> The real vector space V with the complex structure has a canonical orientation.

<u>Proof:</u> Let $v_1 \in V$, $v_1 \neq 0$. Then let $v_2 = Jv_1$.

<u>Claim:</u> v_1 and v_2 are linearly independent.

<u>Proof.</u> Let $a_1 v_1 + a_2 v_2 = 0$, for some $a_1, a_2 \in \mathbb{R}$.

Then $J(a_1 v_1 + a_2 v_2) = J(0) = 0$

$$a_1 J v_1 + a_2 J v_2 = 0 \Rightarrow a_1 v_2 + a_2 J^2 v_1 = 0$$

$$\Rightarrow \boxed{-a_2 v_1 + a_1 v_2 = 0.}$$

$$\begin{aligned} a_1 v_1 + a_2 v_2 &= 0 \quad / a_2 \\ -a_2 v_1 + a_1 v_2 &= 0 \quad / a_1 \\ \hline + \end{aligned}$$

$(a_1^2 + a_2^2) v_2 = 0$ since $v_1 \neq 0$, $v_2 \neq 0$

$\Rightarrow a_1^2 + a_2^2 = 0 \Rightarrow a_1 = a_2 = 0$.

If $\dim_{\mathbb{R}} V = 2n > 2$ then choose $v_3 \in V$, which is <u>not</u> in the linear span of v_1 an

v_2. Let $v_4 = \bar{\jmath} v_3$.

Claim $\{v_1, v_2, v_3, v_4\}$ is $\mathbb{R}$-linearly independent

Proof: Suppose that $a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = 0$ for some $a_i \in \mathbb{R}$, $i = 1, \ldots, 4$. Take $\bar{\jmath}$ of this expression to get the system:

$$\begin{array}{rl} a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = 0 & / \, a_3 \\ + \; a_1v_2 - a_2v_1 + a_3v_4 - a_4v_3 = 0 & / \, -a_4 \\ \hline \end{array}$$

$$(a_1a_3 + a_2a_4)v_1 + (-a_1a_4 + a_2a_3)v_2 + (a_3^2 + a_4^2)v_3 = 0$$

$\Rightarrow a_3^2 + a_4^2 = 0 \Rightarrow a_3 = a_4 = 0$

$\Rightarrow a_1v_1 + a_2v_2 = 0 \Rightarrow a_1 = a_2 = 0$ ◣

This way we see that the set

$\{v_1, \underbrace{\bar{\jmath}v_1}_{v_2}, v_3, \underbrace{\bar{\jmath}v_3}_{v_4}, \ldots, v_{2n-1}, \underbrace{\bar{\jmath}v_{2n-1}}_{v_{2n}}\}$ is linear $\mathbb{R}$-independent.

Claim: $\{v_1, v_3, \ldots, v_{2n-1}\}$ is a $\mathbb{C}$-basis for V.

Proof: If $v \in V$ then $v = \sum_{i=1}^{2n} a_i v_i$, $a_i \in \mathbb{R}$.

$v = a_1v_1 + a_2v_2 + \cdots + a_{2n-1}v_{2n-1} + a_{2n}v_{2n}$

$v_2 = J v_1 = i v_1$

$\Rightarrow v = a_1 v_1 + i a_2 v_1 + \dots + a_{2n-1} v_{2n-1} + i a_{2n} v_{2n-1}$

$= (a_1 + i a_2) \underline{v_1} + \dots + (a_{2n-1} + i a_{2n}) \underline{v_{2n-1}}$

$\Rightarrow \{v_1, v_3, \dots, v_{2n-1}\}$ spans V as a cx. vector space.

Let $\{u_1, u_2 = J u_1, \dots, u_{2n-1}, u_{2n} = J u_{2n-1}\}$ be another such basis for V.

__Claim:__ $\{v_i\}$ and $\{u_i\}$ induce the same orientation.

__Proof:__ $\{v_1, v_3, \dots, v_{2n-1}\} = \beta$ and $\{u_1, u_2, \dots, u_{2n-1}\} = \beta'$ are two complex basis for the complex vector space V. Let $A = [I]_{\beta}^{\beta'} \in \mathbb{C}^{n \times n}$

Let $A = (a_{ij})$ and set $\alpha_{ij} = \begin{pmatrix} Re(a_{ij}) & -Im(a_{ij}) \\ Im(a_{ij}) & Re(a_{ij}) \end{pmatrix}$

The base change matrix from $\{v_1, \dots, v_{2n}\}$ to $\{u_1, \dots, u_{2n}\}$ is the $2n \times 2n$ real matrix obtained from A by replacing each a_{ij} with 2×2-real matrix α_{ij}.

Ex: $\mathbb{C} \xrightarrow{\;a\;} \mathbb{C}$, $a = b + ic$

$\mathbb{R}^2 \xrightarrow[\;A\;]{} \mathbb{R}^2$ $\quad \operatorname{Re} a = b, \ \operatorname{Im} a = c.$

$$a \cdot z = (b+ic)(x+iy) = (bx - cy) + i(cx+by)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \overset{}{\underset{A}{\longmapsto}} \begin{bmatrix} bx - cy \\ cx + by \end{bmatrix} \qquad A = \begin{bmatrix} b & -c \\ c & b \end{bmatrix}$$

<u>must show:</u> $\det A > 0.$

In $n=1$ case, $\det A = b^2 + c^2 > 0.$

$$\underline{A_{\mathbb{C}}} = (a_{ij}) \longleftrightarrow A = (\alpha_{ij})$$

$$\lambda = \mu + i\epsilon \longleftrightarrow \begin{pmatrix} \mu & -\epsilon \\ \epsilon & \mu \end{pmatrix}$$

$$\begin{array}{ccc} A_{\mathbb{C}} & & A \\ \wr\downarrow & \longleftrightarrow & \wr\downarrow \\ \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & \underbrace{\det A_{\mathbb{C}}}_{\theta} \end{pmatrix} & \underset{n\times n}{\longleftrightarrow} & \begin{pmatrix} 1 & 0 & & & 0 \\ 0 & 1 & & & \\ & & \ddots & & \\ & & & \operatorname{Re}\theta & -\operatorname{Im}\theta \\ 0 & & & \operatorname{Im}\theta & \operatorname{Re}\theta \end{pmatrix} \end{array}$$

So, $\det A = |\det A_{\mathbb{C}}|^2 > 0.$

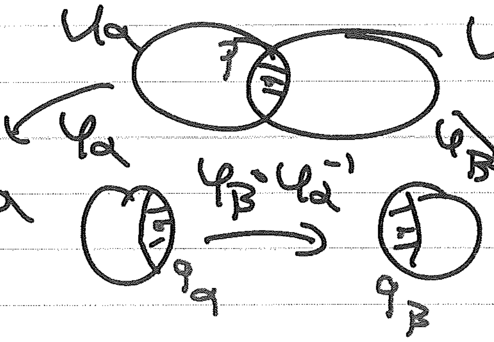
Orientations on Manifolds

$U \subseteq \mathbb{R}^n$ open subset, $T_* U = U \times \mathbb{R}^n$

An orientation on U is a choice of an orientation on the $\mathbb{R}^n$ part of $T_* U$.

Let $\{\varphi_\alpha : U_\alpha \to V_\alpha\}$ be an atlas for a smooth manifold M. Then $T_* U_\alpha \cong T_* V_\alpha = V_\alpha \times \mathbb{R}^n$

Put an orientation on each V_α. If each transition function



if the linear map

$D(\varphi_\beta \circ \varphi_\alpha^{-1})_{(q_\alpha)} : T_{q_\alpha} V_\alpha \longrightarrow T_{q_\beta} V_\beta$

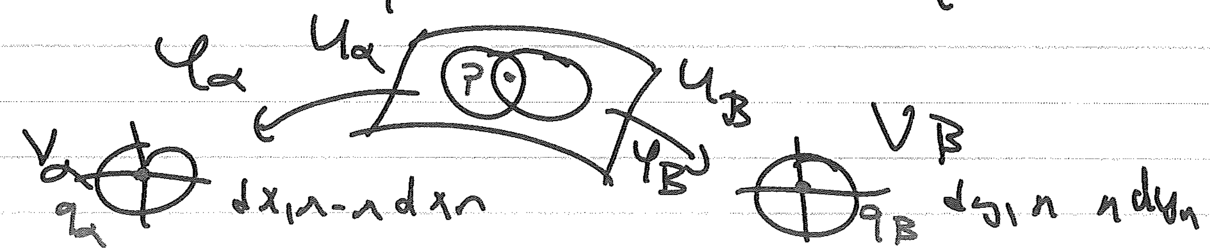
preserves the orientation for all α, β and $q_\alpha = \varphi_\alpha(p)$, $p \in U_\alpha \cap U_\beta$, then we say that the orientations on V_α's put an orientation on M. In this case, we say that M is orientable. Moreover, each choice of orientation makes M an oriented manifold.

If M does <u>not</u> admit any orientation then we say that M is <u>not</u> orientable.

<u>Proposition:</u> A smooth manifold is orientable if and only if there is an n-form ω on M ($\dim M = n$) so that $\omega(p) = f\, dx_1 \wedge \dots \wedge dx_n$ is not zero at any $p \in M$.

<u>Proof.</u> Suppose such ω exists. For any point $p \in M$ and any basis $\{v_1, \dots, v_n\}$ for T_pM we check if $\omega(p)(v_1, \dots, v_n) > 0$. If it is $+$ choose this basis as the orientation at p. This orients the smooth manifold.

If M is oriented then we can choose nowhere zero n-form ω on M as follows:

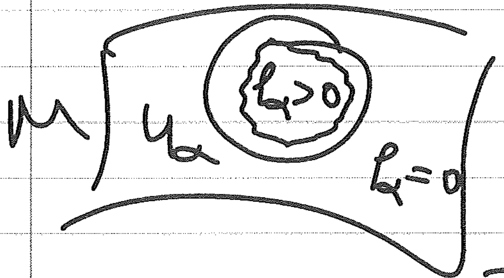


$$\overrightarrow{L = D(\varphi_\beta \circ \varphi_\alpha^{-1})(q_\alpha)}$$

Choose x_i's so that $(dx_1 \wedge \ldots \wedge dx_n)(v_1, \ldots, v_n) > 0$.
Since transition functions preserve local orientation we see that

$$L^*(dy_1 \wedge \ldots \wedge dy_n) = \lambda\, dx_1 \wedge \ldots \wedge dx_n, \quad \lambda > 0$$

Choose a partition of unity $\{\rho_\alpha : U_\alpha \to \mathbb{R}\}$
$\rho_\alpha \geq 0$, $\sum_\alpha \rho_\alpha(p) = 1$, which is a finite sum, and $\operatorname{supp}(\rho_\alpha) \subseteq U_\alpha$.



$$\omega = \sum_\alpha \rho_\alpha \varphi_\alpha^*(dx_1 \wedge \ldots \wedge dx_n)$$

$\omega(v_1, \ldots, v_n) > 0$.
$T_pM = \operatorname{span}\langle v_1, \ldots, v_n \rangle$, oriented basis.

■

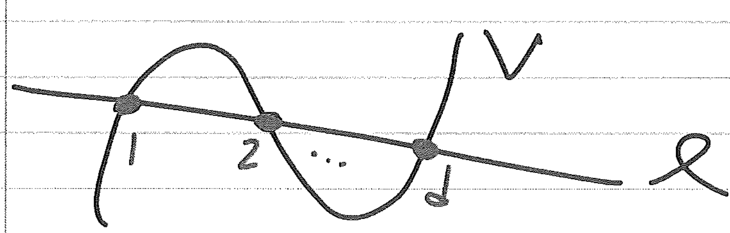
__Ex__: $\mathbb{C}^n$, $V \subseteq \mathbb{C}^n$; $V = (f = 0)$, $f : \mathbb{C}^n \to \mathbb{C}$ poly. map

Assume $0 \in \mathbb{C}$ is a regular value for f, then V is a $n-1$-dim'l complex submanifold.
Let l be a complex line in $\mathbb{C}^n$. Then $V \cap l$ has at most d-points, where $d = \deg f$.

$\ell = \mathbb{C}$, $V: f(z_1, \ldots, z_n) = 0$

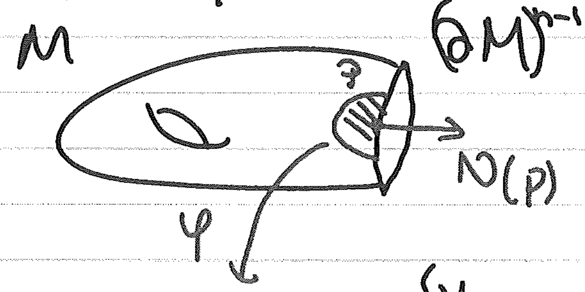
$\ell: z_2 = z_3 = \cdots = z_n = 0$

$V \cap \ell = ?$ $f(z_1, 0, \ldots, 0) = 0 \Rightarrow$ This equation has d solutions (counted with multiplicity)

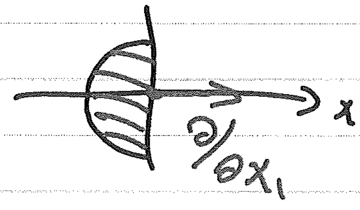


<u>Theorem</u> (Stokes' Theorem)

Let M be a compact smooth oriented manifold. Then ∂M is a smooth orientable manifold of dimension $n-1$.



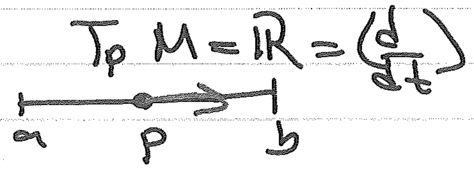
$T_p \partial M = \langle v_1, \ldots, v_{n-1} \rangle$ basis



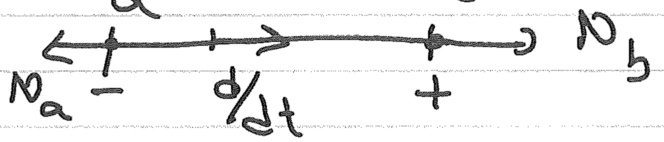
$\{v_1, \ldots, v_{n-1}\}$ gives the right orientation to ∂M at p iff $\{\vec{N}(p), v_1, \ldots, v_{n-1}\}$ gives the right orientation to M at p.

In this case, for any $n-1$-form ω on M we have $\int_{M^n} d\omega = \int_{\partial M} \omega$.

Example: $M = [a,b] \subseteq \mathbb{R}$



$\partial M = \{ \bar{a}, \overset{+}{b} \}$



$\omega = f$ 0-form on $M = [a,b]$

$d\omega = f'(t)\, dt$ 1-form

$$\int_M d\omega = \int_{[a,b]} f'(t)\, dt = f(b) - f(a) = \int_{\partial M = \{a^-, b^+\}} f$$

Math 709, 13,14

Note Title 25.02.2020

Disk ve Kürenin Hacimleri

$$\mathrm{Vol}(D^n(r)) = \int_{D^n(r)} dx_1 \wedge \cdots \wedge dx_n = \int_{D^n(r)} dx_1 \cdots dx_n$$

$$D^n(r) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum x_i^2 \leq 1\}$$

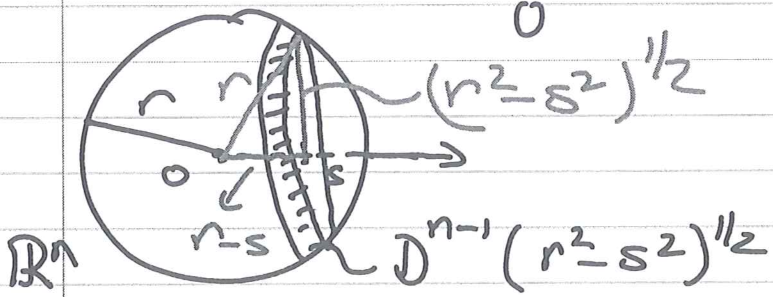
For any integer $n > 0$ let $\mathrm{Vol}(D^n(r)) = r^n \cdot A_n$.

So $A_n = \dfrac{\mathrm{Vol}(D^n(r))}{r^n}$.

$$\mathrm{Vol}(D^1(r)) = \mathrm{Length}([-1,1]) = 2 \Rightarrow A_1 = \frac{2}{1^1} = 2.$$

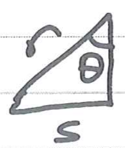
$$\mathrm{Vol}(D^2(r)) = \pi r^2 \Rightarrow A_2 = \frac{\pi r^2}{r^2} = \pi.$$

$$\mathrm{Vol}(D^n(r)) = 2\int_0^r \underbrace{\mathrm{Vol}(D^{n-1}(\sqrt{r^2-s^2}))}_{=\,(r^2-s^2)^{\frac{n-1}{2}} A_{n-1}}\, ds$$



$$\mathrm{Vol}(D^n(r)) = 2A_{n-1}\int_0^r \left(\sqrt{r^2-s^2}\right)^{n-1} ds$$

Let $s = r\sin\theta$



$\sin\theta = \dfrac{s}{r}$ $\qquad$ s: 0 to r $\longrightarrow$ θ: 0 to $\pi/2$

$$r^2 - s^2 = (\cos^2\theta) r^2$$

$$ds = r\cos\theta\, d\theta$$

Hence, we get

$$\mathrm{Vol}(D^n(r)) = 2A_{n-1}\int_0^{\pi/2} (\cos\theta)^n r^n \, d\theta$$

$$= 2r^n A_{n-1} \int_0^{\pi/2} \cos^n\theta \, d\theta$$

Let $B_n \doteq \int_0^{\pi/2} \cos^n\theta \, d\theta$.

$$B_1 = \int_0^{\pi/2} \cos\theta \, d\theta = \sin\theta \Big|_0^{\pi/2} = 1.$$

$$B_2 = \int_0^{\pi/2} \cos^2\theta \, d\theta = \pi/4.$$

For $n \geq 2$, let $v = (\cos\theta)^{n-1}$; $du = \cos\theta \, d\theta$

$u = \sin\theta$

$dv = (n-1)\cos^{n-2}\theta \, (-\sin\theta)$

$$B_n = \int_0^{\pi/2} \cos^n\theta \, d\theta = \int_0^{\pi/2} v \, du$$

$$= uv\Big|_0^{\pi/2} - \int_0^{\pi/2} u \, dv$$

$$= (\cos\theta)^{n-1}\sin\theta\Big|_0^{\pi/2} + \int_0^{\pi/2} \sin^2\theta \cos^{n-2}\theta \, (n-1)\, d\theta$$

$$= 0 + \int_0^{\pi/2} (1-\cos^2\theta)\cos^{n-2}\theta \, (n-1)\, d\theta$$

$$= \left[\int_0^{\pi/2} \cos^{n-2}\theta \, d\theta - \int_0^{\pi/2} \cos^n\theta \, d\theta\right](n-1)$$

$$B_n = (n-1)B_{n-2} - (n-1)B_n$$

$$nB_n = (n-1)B_{n-2} \Rightarrow B_n = \frac{n-1}{n} B_{n-2}.$$

$$B_{2n+1} = \frac{(2^n n!)^2}{(2n+1)!} \quad \text{and} \quad B_{2n} = \frac{(2n)!}{(2^n n!)^2}\frac{\pi}{2}$$

We also have $A_n / A_{n-1} = 2B_n$.

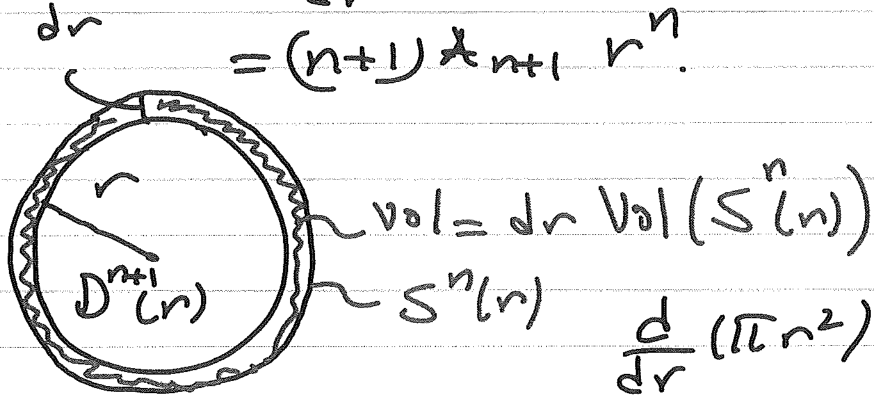
Using this we get

$$A_{2n+1} = \frac{2^{n+1}\pi^n}{1\cdot 3 \cdots (2n+1)} \quad \text{and} \quad A_{2n} = \frac{\pi^n}{n!}$$

So, $\mathrm{Vol}(D^n(r)) = A_n r^n$ is computed.

On the other hand, we have

$$\begin{aligned}\mathrm{Vol}(S^n(r)) &= \frac{d}{dr}\mathrm{Vol}(D^{n+1}(r)) \\ &= \frac{d}{dr}(A_{n+1} r^{n+1}) \\ &= (n+1)A_{n+1} r^n.\end{aligned}$$



$$\frac{d}{dr}(\pi r^2) = 2\pi r$$

Example: $\mathrm{Vol}(D^4(r)) = \frac{\pi^2 r^4}{2}$, $\mathrm{Vol}(S^3(r)) = 2\pi^2 r^3$.

Example: $\mathbb{R}^n \setminus \{0\}$, $\omega_{S^{n-1}} \in \Omega^{n-1}(\mathbb{R}^n \setminus \{0\})$

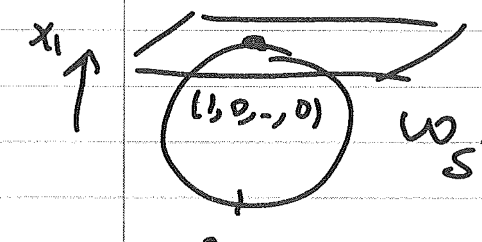
$$\omega_{S^{n-1}} \doteq \sum_{i=1}^{n} (-1)^{i-1} x_i \frac{dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n}{(x_1^2 + \cdots + x_n^2)^{n/2}}$$

$d\omega_{S^{n-1}} = 0$ (Exercise)

$A \in SO(n)$, $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

Exercise: $A^* \omega_{S^{n-1}} = \omega_{S^{n-1}}$, so that $\omega_{S^{n-1}}$ is $SO(n)$ invariant.

Let's compute this for on the basis vectors $\{\frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\}$ of $T_{(1,0,\ldots,0)} S^{n-1}$.



$$\omega_{S^{n-1}}\left(\frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right) = 1.$$

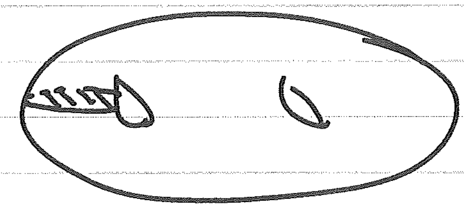
$$\int_{S^{n-1}} \omega_{S^{n-1}} = \text{Vol}(S^{n-1}) = n A_n$$

Now define $\omega_{0,\mathbb{R}^n} = \dfrac{\omega_{S^{n-1}}}{n A_n}$.

Then $\displaystyle\int_{S^{n-1}} \omega_{0,\mathbb{R}^n} = 1.$

Soon, we'll see that $\omega_{0,\mathbb{R}^n}$ is the only nontrivial $n-1$-form on $\mathbb{R}^n \setminus \{0\}$.

<u>Remark:</u> Let $M \subseteq \mathbb{R}^n$ be an $n-1$ dimensional smooth closed oriented manifold, so that $0 \notin M$.



$M^{n-1} = \partial V^n$ for some smooth compact manifold V^n.

$$\int_M \omega_{0,\mathbb{R}^n} = \frac{1}{nA_n} \int_M \omega_{S^{n-1}}$$

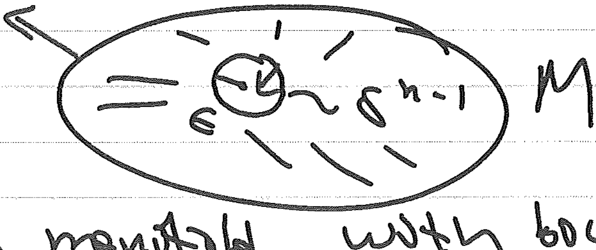
<u>If $0 \notin V^n$</u>, then from Stokes' Theorem

$$\int_M \omega_{S^{n-1}} = \int_{\partial V^n} \omega_{S^{n-1}} = \int_{V^n} \underbrace{d\omega_{S^{n-1}}}_{=0} = 0.$$

$= 0$ since $\omega_{S^{n-1}}$ is closed

So $\int_M \omega_{0,\mathbb{R}^n} = 0$.

<u>If $0 \in V^n$</u>, then choose a small sphere S_ϵ^{n-1} around 0.



$V^n \setminus \overset{\circ}{S}{}_\epsilon^{n-1}$ = compact smooth manifold with boundary

$\partial(V^n \setminus S_\epsilon^{n-1}) = M \,\dot{\cup}\, (-S_\epsilon^{n-1})$. In particular, $0 \notin V^n \setminus S_\epsilon^{n-1}$, and thus

$$0 = \int_{V^n \setminus S_\epsilon^{n-1}} \overbrace{d\omega_{0,\mathbb{R}^n}}^{=0} = \int_{\partial(V^n \setminus S_\epsilon^{n-1}) = M \setminus (-S_\epsilon^{n-1})} \omega_{0,\mathbb{R}^n}$$

$$= \int_M \omega_{0,\mathbb{R}^n} - \int_{S_\epsilon^{n-1}} \omega_{0,\mathbb{R}^n}$$

$$\Rightarrow \int_M \omega_{0,\mathbb{R}^n} = \int_{S_\epsilon^{n-1}} \omega_{0,\mathbb{R}^n}$$

In particular, if we take $M = S_1^{n-1}$, then

$$\underline{1} = \int_{S_1^{n-1}} \omega_{0,\mathbb{R}^n} = \int_{S_\epsilon^{n-1}} \omega_{0,\mathbb{R}^n}$$

Hence, $\int_M \omega_{0,\mathbb{R}^n} = 1$.

Summary: $M^{n-1} = \partial V^n$, $V^n \subseteq \mathbb{R}^n$ smooth compact manifold.

Then

$$\int_M \omega_{0,\mathbb{R}^n} = \begin{cases} 0 & \text{if } 0 \notin V^n \\ 1 & \text{if } 0 \in V^n. \end{cases}$$

$\omega_{0,\mathbb{R}^n}$ is called the "linking form" of the origin in $\mathbb{R}^n$.

Special Forms on Complex Manifolds:

$\mathbb{C}^n = \mathbb{R}^{2n}$ $\quad z_1, \dots, z_n$, $z_k = x_k + i y_k$

$z_k \quad x_k, y_k$

$dz_k = dx_k + i\, dy_k$. Consider the 2-form on $\mathbb{C}^n = \mathbb{R}^{2n}$, given by

$$\omega = \frac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k = \sum_{k=1}^{n} dx_k \wedge dy_k,$$

where $\bar{z}_k = x_k - i y_k$ and $d\bar{z}_k = dx_k - i\, dy_k$.

Exercise: $\omega \in \Omega^2(\mathbb{C}^n) = \Omega^2(\mathbb{R}^{2n})$.

For any $0 \leq l \leq n$ (integer) we have

$$\omega^l = \left(\sum_{k=1}^{n} dx_k \wedge dy_k\right)^l = \left(\frac{i}{2}\sum_{k=1}^{n} dz_k \wedge d\bar{z}_k\right)^l$$

$$= \left(\frac{i}{2}\right)^l l! \sum_{1 \leq k_1 < k_2 < \dots < k_l \leq n} dz_{k_1} \wedge d\bar{z}_{k_1} \wedge \dots \wedge dz_{k_l} \wedge d\bar{z}_{k_l}$$

Let $V \subseteq \mathbb{C}^n$ be an l-complex dimensional subspace of $\mathbb{C}^n$. Let $(w_1, \dots, w_l)$ be a linear coordinate system on V: $w_i : V \to \mathbb{C}$ linear map.

Let $L : V \to \mathbb{C}^n$ be the inclusion map given by the expression

(annotations above: w_j over V, z_j over $\mathbb{C}^n$)

$$(z_1, \dots, z_n) = L(w_1, \dots, w_\ell)$$
$$= (a_{11} w_1 + \dots + a_{\ell 1} w_\ell,\ a_{12} w_1 + \dots + a_{\ell 2} w_\ell,$$
$$\dots,\ a_{1n} w_1 + \dots + a_{\ell n} w_\ell)$$

$$\overline{z}_k = \overline{a_{1k}}\,\overline{w}_1 + \overline{a_{2k}}\,\overline{w}_2 + \dots + \overline{a_{\ell k}}\,\overline{w}_\ell, \quad k = 1, \dots, n.$$

$A = (a_{ij})$ a complex $\ell \times n$-matrix.

Then $L^*(dz_j) = a_{1j}\, dw_1 + \dots + a_{\ell j}\, dw_\ell$, and

$L^*(d\overline{z}_j) = \overline{a_{1j}}\, d\overline{w}_1 + \dots + \overline{a_{\ell j}}\, d\overline{w}_\ell$.

Thus we can compute

$$L^*(dz_{k_1} \wedge d\overline{z}_{k_1} \wedge \dots \wedge dz_{k_\ell} \wedge d\overline{z}_{k_\ell})$$

$$= \overline{\det(A_{k_1 \dots k_\ell})}\, \det(A_{k_1 \dots k_\ell})\, dw_1 \wedge d\overline{w}_1 \wedge \dots \wedge dw_\ell \wedge d\overline{w}_\ell$$

$$= \overbrace{\|\det(A_{k_1 \dots k_\ell})\|^2}^{0 \le} dw_1 \wedge d\overline{w}_1 \wedge \dots \wedge dw_\ell \wedge d\overline{w}_\ell,$$

where $A_{k_1 \dots k_\ell}$ is the submatrix of $A = (a_{ij})$ consisting of the rows $k_1, \dots, k_\ell$.

$L^*(\omega^l) = C_V \left(\frac{i}{2}\right)^l l!\, dw_1 \wedge d\bar{w}_1 \wedge \dots \wedge dw_l \wedge d\bar{w}_l,$

where $C_V > 0$ is a constant.

sum of the positive determinants

$w_1 = u_1 + iv_1, \dots, w_l = u_l + iv_l$

V has a canonical orientation given by

$$\left\{\frac{\partial}{\partial u_1}, i\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_l}, i\frac{\partial}{\partial u_l}\right\} = \left\{\frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1}, \dots, \frac{\partial}{\partial u_n}, \frac{\partial}{\partial v_n}\right\}$$

$$L^*(\omega^l)\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1}, \dots, \frac{\partial}{\partial u_l}, \frac{\partial}{\partial v_l}\right)$$

$$= C_V\left(\frac{i}{2}\right)^l l!\, \underline{dw_1 \wedge d\bar{w}_1 \wedge \dots \wedge dw_l \wedge d\bar{w}_l}\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1}, \dots, \frac{\partial}{\partial u_n}, \frac{\partial}{\partial v_n}\right)$$

$$= C_V\left(\frac{i}{2}\right)^l l!\left(\frac{2}{i}\right)^l du_1 \wedge dv_1 \wedge \dots \wedge du_l \wedge dv_l \left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1}, \dots, \frac{\partial}{\partial u_n}, \frac{\partial}{\partial v_n}\right)$$

$$= C_V\, l! \cdot \underline{1}$$

$$= C_V\, l! > 0.$$

<u>Conclusion</u>: If M is an complex l-dim'l smooth manifold of $\mathbb{C}^n$ then the restriction of ω^l to M evaluates positively at any

complex oriented tangent space T_pM of M.
In particular, if $U \subseteq M$ is an open subset with compact closure then
$$\int_U \omega^l > 0.$$

Note Title

$\mathbb{C}^n$, $\omega = \frac{i}{2}\sum_{i=1}^n dz_i \wedge d\bar{z}_i$

$= \sum_{i=1}^n dx_i \wedge dy_i$, $z_j = x_j + i y_j$

$V \subseteq \mathbb{C}^n$ l dim'l cx. subspace

$w_1, \ldots, w_l$, $w_j = u_j + i v_j$

$L : V \hookrightarrow \mathbb{C}^n$ inclusion map

$L^* \omega \left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1}, \ldots, \frac{\partial}{\partial u_l}, \frac{\partial}{\partial v_l}\right) > 0.$

Corollary: $\mathbb{C}^n$ has no closed smooth complex submanifold of positive dimension.

Proof: Let $M^l \subseteq \mathbb{C}^n$ closed complex submanifold of dimension $l \geq 0$. Then, by above

$\int_{M^l} \omega^l > 0.$ On the other hand, the form ω can be written as

$\omega = \sum_{i=1}^n dx_i \wedge dy_i = d\left(\sum_{i=1}^n x_i dy_i\right) = d\eta$, where

$\eta = \sum_{i=1}^n x_i dy_i$. So, ω is exact.

Hence, $\omega^l = \omega \wedge \omega^{l-1} = d\eta \wedge \omega^{l-1} = d(\eta \wedge \omega^{l-1})$, (with $d\omega^{l-1} = 0$)

provided that $l>0$. However, in this case

$$0<\int_M \omega^l=\int_M d(\eta\wedge\omega^{l-1}) \underset{\text{Stokes'}}{=} \int \eta\wedge\omega^{l-1}=0, \text{ a}$$

$\partial M=\emptyset$

contradiction.

Hence, $l=\dim M$ must be zero. ∎

Example: Hence, $\mathbb{CP}^n$, $n\geq 1$ and $S^1\times S^3$ (with its complex structure) do not admit any embedding into some $\mathbb{C}^N$.

Forms on Complex Projective Space

$S^2=\mathbb{CP}^1$ $\quad x,y,z$

$\cap$

$\mathbb{R}^3$ $\quad \omega=x\,dy\wedge dz+y\,dz\wedge dx+z\,dx\wedge dy$

$\omega\in\Omega^2(\mathbb{R}^3)$, $\int_{S^2}\omega=4\pi$.

$\varphi^{-1}:\mathbb{R}^2\to S^2\setminus\{(0,0,-1)\}\subseteq\mathbb{R}^3$

$$\varphi^{-1}(r,s)=\Big(\underbrace{\frac{2r}{1+r^2+s^2}}_{x},\underbrace{\frac{2s}{1+r^2+s^2}}_{y},\underbrace{\frac{1-r^2-s^2}{1+r^2+s^2}}_{z}\Big),$$

$$x=\frac{2r}{1+r^2+s^2},\quad y=\frac{2s}{1+r^2+s^2},\quad z=\frac{1-r^2-s^2}{1+r^2+s^2}$$

$$dx = \frac{2(1-r^2+s^2)dr - 4rs\,ds}{(1+r^2+s^2)^2}$$

$$dy = \frac{2(1-s^2+r^2)ds - 4rs\,dr}{(1+r^2+s^2)^2}, \quad d\rho = -\frac{4r\,dr + 4s\,ds}{(1+r^2+s^2)^2}$$

$$\Rightarrow (\Phi^{-1})^* \omega = 4 \frac{dr \wedge ds}{(1+r^2+s^2)^2} = 2i \frac{dz \wedge d\bar{z}}{(1+\|z\|^2)^2},$$

where $z = r + is$.

$w = \frac{1}{z}$ $\quad \mathbb{CP}^1 = \{[z_0 : z_1]\}$ $\quad \frac{z_1}{z_0} = z, \frac{z_0}{z_1} = w$

$$(\Phi^{-1})^* \omega = 2i \frac{dw \wedge d\bar{w}}{(1+\|w\|^2)^2}.$$

$\frac{1}{4}\omega$ is called the Fubini–Study 2-form on $\mathbb{CP}^1$ and denoted as ω_{FS}.

$$\int_{\mathbb{CP}} \omega_{FS} = \pi.$$

~ o ~

$f : \mathbb{C}^n \to \mathbb{C}$ smooth function.

$$\partial f \doteq \sum_{i=1}^{n} \frac{\partial f}{\partial z_i} dz_i, \quad \bar{\partial} f = \sum_{i=1}^{n} \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i$$

<u>Claim</u>: $\omega_{FS} = \frac{i}{2} \frac{dz \wedge d\bar{z}}{(1+\|z\|^2)^2} = \frac{i}{2} \partial\bar{\partial} \log(1+\|z\|^2)$.

<u>Proof</u>: $\bar{\partial} \log(1+\|z\|^2) = \frac{\partial}{\partial \bar{z}} \log(1+z\bar{z})\, d\bar{z}$

So, $\bar\partial \log(1+\|z\|^2) = \frac{z}{1+z\bar z}\, d\bar z$, hence

$$\partial\bar\partial \log(1+\|z\|^2) = \partial\left(\frac{z}{1+z\bar z}\, d\bar z\right)$$
$$= \frac{\partial}{\partial z}\left(\frac{z}{1+z\bar z}\right) dz\wedge d\bar z$$
$$= \frac{1\cdot(1+z\bar z) - z\bar z}{(1+z\bar z)^2}\, dz\wedge d\bar z.$$
$$= \frac{1}{(1+\|z\|^2)^2}\, dz\wedge d\bar z.$$

$$\omega_{FS} = \frac{i}{2}\partial\bar\partial\log(1+\|z\|^2) \qquad z = z_1/z_0$$
$$= \frac{i}{2}\partial\bar\partial\log\left(\frac{\|z_0\|^2+\|z_1\|^2}{\|z_0\|^2}\right)$$
$$= \frac{i}{2}\partial\bar\partial\log(\|z_0\|^2+\|z_1\|^2) - \underbrace{\frac{i}{2}\partial\bar\partial\log\|z_0\|^2}_{=0}$$
$$= \frac{i}{2}\partial\bar\partial\log(\|z_0\|^2+\|z_1\|^2)$$

<u>Definition</u>: The Fubini–Study form ω_{FS} on $\mathbb{CP}^n$ is defined as

$$\omega_{FS} \doteq \frac{i}{2}\partial\bar\partial\log(\|z_0\|^2+\|z_1\|^2+\dots+\|z_n\|^2).$$

<u>Example</u>: $n=2$, $\mathbb{CP}^2$, $\omega_{FS} = \frac{i}{2}\partial\bar\partial\log(\|z_0\|^2+\|z_1\|^2+\|z_2\|^2)$

$$\Rightarrow \omega_{FS} = \frac{i}{2} \frac{(1+z_2\bar{z}_2)\,dz_1\wedge d\bar{z}_1 + (1+z_1\bar{z}_1)\,dz_2\wedge d\bar{z}_2}{(1+z_1\bar{z}_1+z_2\bar{z}_2)^2}$$

$$+ \frac{i}{2} \frac{z_1\bar{z}_2\,d\bar{z}_1\wedge dz_2 + z_2\bar{z}_1\,d\bar{z}_2\wedge dz_1}{(1+z_1\bar{z}_1+z_2\bar{z}_2)^2}$$

$z_j = x_j + i y_j$, $j = 1, 2$.

$$\omega_{FS}\wedge\omega_{FS} = 2\left(\frac{i}{2}\right)^2 \frac{dz_1\wedge d\bar{z}_1\wedge dz_2\wedge d\bar{z}_2}{(1+z_1\bar{z}_1+z_2\bar{z}_2)^3}$$

$$= \frac{2}{(1+x_1^2+y_1^2+x_2^2+y_2^2)^3}\, dx_1\wedge dy_1\wedge dx_2\wedge dy_2$$

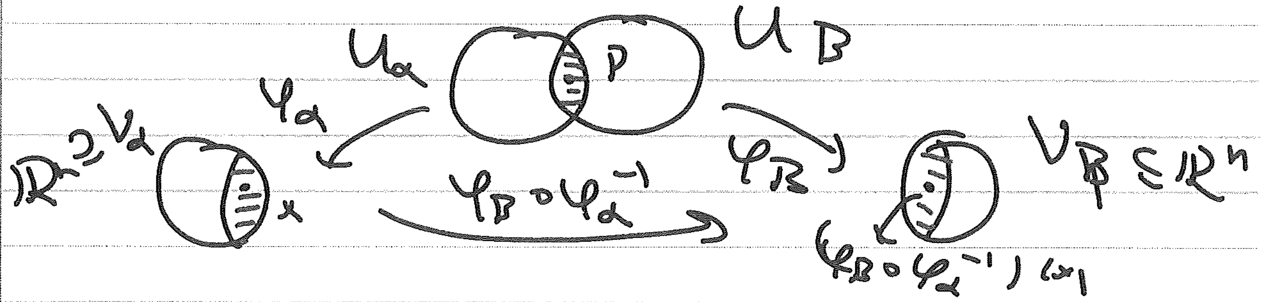
$$\int_{\mathbb{CP}^2} \omega_{FS}\wedge\omega_{FS} = \int_{\mathbb{R}^4} \omega_{FS}\wedge\omega_{FS} \stackrel{?}{=} \pi^2.$$

Vector Bundles:

M smooth manifold with an atlas $\{\varphi_\alpha : U_\alpha \to V_\alpha\}$.

$$M = \bigcup_\alpha U_\alpha = \dot{\bigcup_\alpha} V_\alpha \Big/ x \sim (\varphi_\beta \circ \varphi_\alpha^{-1})(x)$$

$U_\alpha \subseteq M$, $V_\alpha \subseteq \mathbb{R}^n$



$$T_* M = \bigcup_\alpha T_* U_\alpha = \dot{\bigcup_\alpha} T_* V_\alpha = \dot{\bigcup_\alpha} (V_\alpha \times \mathbb{R}^n) \Big/ (x,v) \sim ((\varphi_\beta \circ \varphi_\alpha^{-1})(x), w)$$

when $w = D(\varphi_\beta \circ \varphi_\alpha^{-1})_{(x)}(v)$

$$T^* M = \bigcup_\alpha T^* U_\alpha = \dot{\bigcup_\alpha} T^* V_\alpha = \dot{\bigcup_\alpha} V_\alpha \times (\mathbb{R}^n)^* \Big/ \sim$$

$$(x, D(\varphi_\beta \circ \varphi_\alpha^{-1})^*_{(x)}(w)) \sim ((\varphi_\beta \circ \varphi_\alpha^{-1})(x), w)$$

Definition: Let $P : E^{m+k} \to M^m$ be a smooth map of smooth manifolds satisfying the following conditions:

1) For any $p \in M$, $E_p = P^{-1}(p)$ is a k-dim'l real vector space.

2) There is an open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of M so that

i) For each $\alpha \in \Lambda$ there is a diffeomorphism

$$\phi_\alpha : P^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^k$$

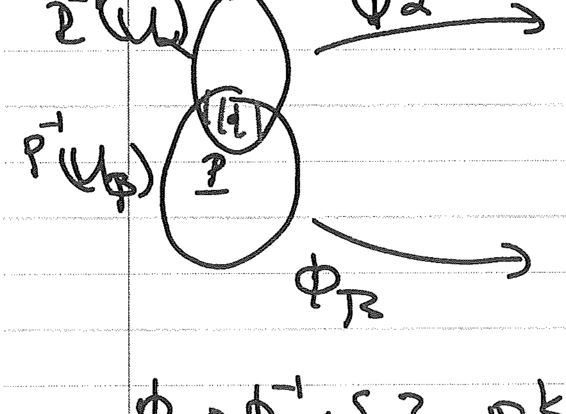
ii) For each $\alpha \in \Lambda$ and $p \in U_\alpha$ the restriction

map $\phi_\alpha|_{E_p} : E_p \longrightarrow \{p\} \times \mathbb{R}^k$ is a linear isomorphism of real vector spaces.

$E_p = P^{-1}(p)$

In this case, we say that $p : E^{m+k} \longrightarrow M^m$ is a smooth real vector bundle of rank k.

<u>Transition function:</u>



$$\begin{array}{ccc} U_\alpha \times \mathbb{R}^k & & (p, v) \\ \downarrow \phi_\beta \circ \phi_\alpha^{-1} & & \downarrow \\ U_\beta \times \mathbb{R}^k & & (p, (\phi_\beta \circ \phi_\alpha^{-1})(v)) \end{array}$$

$\phi_\beta \circ \phi_\alpha^{-1}|_p : \{p\} \times \mathbb{R}^k \longrightarrow \{p\} \times \mathbb{R}^k$ is a linear isomorphism.

$$U_\alpha \cap U_\beta \longrightarrow GL(k,\mathbb{R})$$
$$p \longmapsto (\phi_\beta \circ \phi_\alpha^{-1}) : \{p\}\times\mathbb{R}^k \longrightarrow \{p\}\times\mathbb{R}^k$$

We write this as

$$\phi_\beta \phi_\alpha^{-1}(p,v) = (p, \psi_{\beta\alpha}(p)(v)) \quad , \psi_{\beta\alpha}(p) \in GL(k,\mathbb{R})$$

$\psi_{\beta\alpha}$ is called a transition function for the vector bundle. Note that they satisfy the cocycle condition $\psi_{\alpha\beta} \circ \psi_{\beta\gamma} = \psi_{\alpha\gamma}$.

<u>Ex</u> $T_\mathbb{C}\mathbb{CP}^1 \cong U_0\times\mathbb{C} \,\dot{\cup}\, U_1\times\mathbb{C} \big/ (z,v)\sim(1/z, -\frac{1}{z^2}v)$

$$\phi_0 : p^{-1}(U_0) \longrightarrow U_0\times\mathbb{C} \qquad (z,v)$$
$$\phi_1 : p^{-1}(U_1) \longrightarrow U_1\times\mathbb{C} \qquad (1/z, -\tfrac{1}{z^2}v)$$

$$\psi_{01} : U_0\cap U_1 \longrightarrow GL(1,\mathbb{C})$$
$$z \longmapsto \left[-\frac{1}{z^2}\right]$$

This examples shows that the above definition can be made for the field $\mathbb{C}$ or even $\mathbb{H}$.

Over $\mathbb{C}$ we get complex vector bundles and over $\mathbb{H}$ we get Quaternion vector bundles.

Remark: If M is a smooth manifold and $\{U_\alpha\}_{\alpha\in\Delta}$ is an open cover let

$\psi_{\alpha\beta}: U_\alpha \cap U_\beta \longrightarrow GL(k,\mathbb{F})$ $(\mathbb{F}=\mathbb{R},\mathbb{C},\mathbb{H})$

be smooth functions satisfying the conditions

i) $\psi_{\alpha\beta}(p) = (\psi_{\beta\alpha}(p))^{-1}$

ii) $\psi_{\alpha\beta} \circ \psi_{\beta\gamma} = \psi_{\alpha\gamma}$, for all α, β, γ, then we obtain a smooth $\mathbb{F}$-vector bundle of rank k as follows:

$$\begin{array}{l} E = \dot{\bigcup\limits_\alpha} (U_\alpha \times \mathbb{F}^k) \Big/ (p,v) \sim (p, \psi_{\alpha\beta}(p)(v)) \\ \downarrow \\ M \end{array}$$

if $p \in U_\alpha \cap U_\beta$.

Operations on Vector Bundles:

Summation: let $E_i \longrightarrow M$, $i=1,2$, be vector bundles of rank k_1 and k_2. Suppose $E_i \to M$ has transition functions $\{\psi^i_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k_i, \mathbb{F})\}$. Then the direct sum $E_1 \oplus E_2 \to M$ of E_1 and E_2 is the vector bundle with transition functions

$$\psi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(k_1+k_2, \mathbb{F})$$

$$p \longmapsto \begin{bmatrix} \psi^1_{\alpha\beta} & 0 \\ 0 & \psi^2_{\alpha\beta} \end{bmatrix}_{(k_1+k_2)\times(k_1+k_2)}$$

so that the direct sum is a vector bundle of rank k_1+k_2.

<u>Determinant Line Bundle:</u>

Let $P : E^{n+k} \to M^n$ be a vector bundle of rank k ($\mathbb{F} = \mathbb{R}, \mathbb{C}$), with transition functions $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(k, \mathbb{F})$.

Consider the composition

$$\det \circ \psi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow k^* = GL(1, \mathbb{F})$$

$\varphi_{\alpha\beta} = \det \circ \psi_{\alpha\beta}$ still satisfy the conditions:

$\varphi_{\alpha\beta} = \det(\Psi_{\alpha\beta})$

i) $\varphi_{\beta\alpha} = \det(\Psi_{\beta\alpha}) = \det(\Psi_{\alpha\beta}^{-1})$

$= \left(\det(\Psi_{\alpha\beta})\right)^{-1} = (\varphi_{\alpha\beta})^{-1}$

ii) $\varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} = \det(\Psi_{\alpha\beta}) \cdot \det(\Psi_{\beta\gamma})$

$= \det(\Psi_{\alpha\beta} \cdot \Psi_{\beta\gamma})$

$= \det(\Psi_{\alpha\gamma})$

$= \varphi_{\alpha\gamma}.$

The rank 1 vector bundle with transition functions $\varphi_{\alpha\beta} = \det(\Psi_{\alpha\beta})$ is called the determinant line bundle of $E \to M$ and denoted by $\det(E) \to M$.

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Note Title

$p: E^{n+k} \to M^n$ smooth $\mathbb{R}^k$-bundle if

$p^{-1}(x) = E_x$ is an k-dim'l $\mathbb{R}$-vector space

$M = \bigcup_\alpha U_\alpha$, $U_\alpha \subseteq M^n$ open and

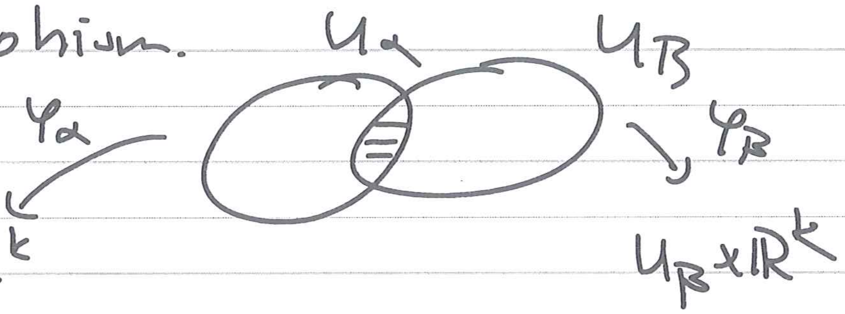
trivializations $\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$

smooth map and for any $x \in U_\alpha$ the

restriction map $\varphi_\alpha|_{p^{-1}(x)} : E_x \to \{x\} \times \mathbb{R}^k$

and $\varphi_\alpha|_{p^{-1}(x)}$ is a $\mathbb{R}$-linear vector space

isomorphism.



$$(U_\alpha \cap U_\beta) \times \mathbb{R}^k \longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^k$$
$$(x, v) \longmapsto (x, (\varphi_\beta \circ \varphi_\alpha^{-1})(x, v))$$

So we get so called transition functions

$$\psi_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow GL(k, \mathbb{R})$$
$$x \longmapsto \psi_{\beta\alpha}(x) = (\varphi_\beta \circ \varphi_\alpha^{-1})(x, \cdot)$$

$\{\psi_{\alpha\beta}\}$ clearly satisfies

1) $\psi_{\alpha\alpha} = id$

2) $\psi_{\gamma\beta} \circ \psi_{\beta\alpha} = (\varphi_\gamma \circ \varphi_\beta^{-1}) \circ (\varphi_\beta \circ \varphi_\alpha^{-1})$

$= \varphi_\gamma \circ \varphi_\alpha^{-1} = \psi_{\gamma\alpha}.$

(Cocycle Condition)

Conversely, if we have an open cover $\{U_\alpha\}$ of M and a collection of smooth maps

$\psi_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow GL(k, \mathbb{R})$ satisfying the conditions (1) and (2) above then we obtain a vector bundle $E \longrightarrow M$ as follows:

$$E = \dot{\bigcup} U_\alpha \times \mathbb{R}^k \Big/ (x, v) \sim (x, \psi_{\beta\alpha}(x)(v))$$

for any $x \in U_\alpha \cap U_\beta$.

<u>Direct Sum of Vector Bundles:</u>

$E_1^{k_1+n} \longrightarrow M^n$, $E_2^{k_2+n} \longrightarrow M^n$ two vector bundles

$E_1 \oplus E_2 \longrightarrow M$ is defined as follows:

If $\psi_{\beta\alpha}^i : U_\alpha \cap U_\beta \longrightarrow GL(k_i, \mathbb{R})$ are the transition functions then the transition functions of $E_1 \oplus E_2$ are as follows

$$x \longmapsto \begin{bmatrix} \psi^1_{\beta\alpha} & \\ & \psi^2_{\beta\alpha} \end{bmatrix}_{(k_1+k_2)\times(k_1\times k_2)}$$

<u>Determinant Bundle</u> $E \longrightarrow M, \{\psi_{\beta\alpha}\}$

$$U_\alpha \cap U_\beta \longrightarrow GL(k,\mathbb{R}) \xrightarrow{\det} GL(1,\mathbb{R}) = \mathbb{R}^*$$

The bundle with transition functions $\{\det(\psi_{\beta\alpha})\}$ is a $\mathbb{R}$-bundle, called the determinant line bundle of $E \to M$.

<u>Remark:</u> Similarly one can define the $\mathbb{F}$ bundles when $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$.

<u>Bundle of Homomorphisms</u>

$E_i \longrightarrow M$ rank k_i, $\mathbb{F}$-bundles.

$(\mathbb{F} = \mathbb{R}, \mathbb{C})$ $\hom(E_1, E_2) \longrightarrow M$ is so that the fiber of any $x \in M$ is $\hom(E_{1x}, E_{2x})$, which is a vector space of dimension $k_1 k_2$.

The transition functions of $\hom(E_1, E_2)$ are defined by

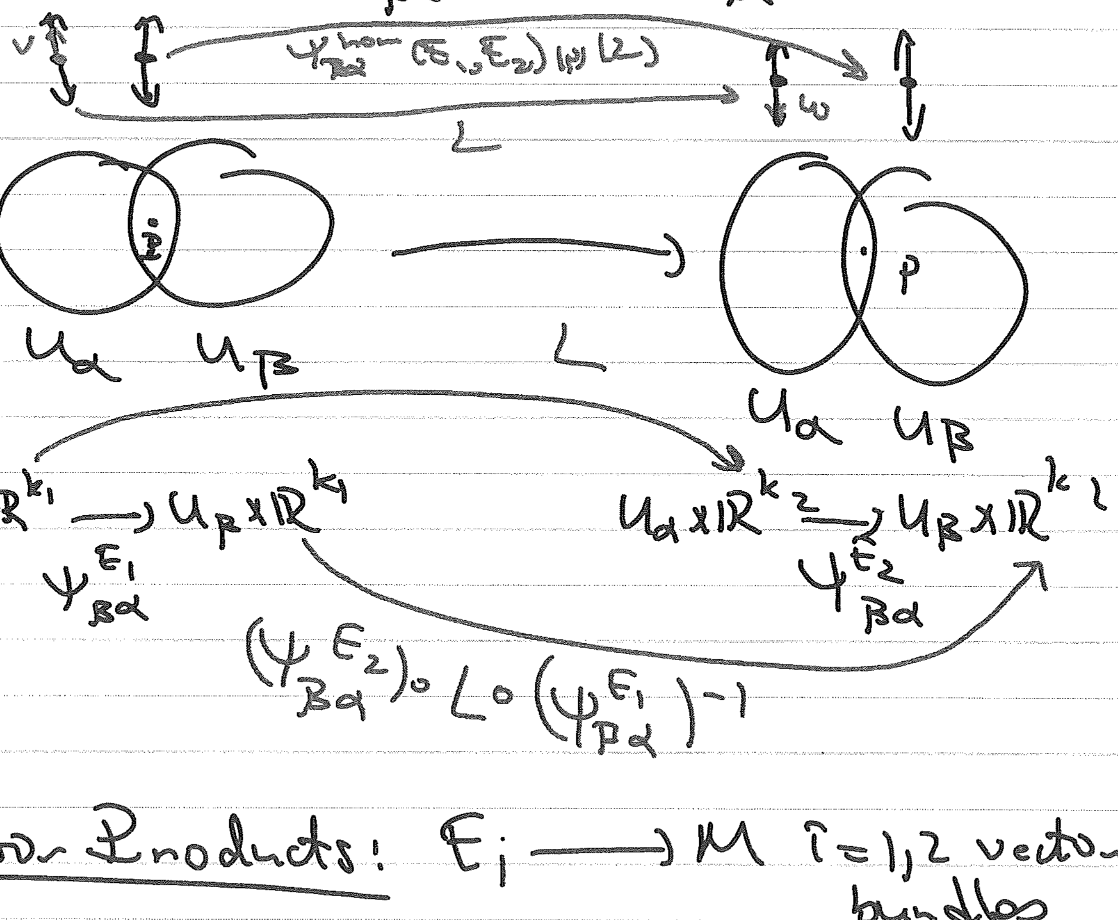
$$\psi_{\beta\alpha}^{\hom(E_1,E_2)} : U_\alpha \cap U_\beta \longrightarrow GL(\hom(\mathbb{R}^{k_1}, \mathbb{R}^{k_2}), \mathbb{R})$$

$$p \longmapsto \left(L \mapsto \psi_{\beta\alpha}^{E_2}(p) \circ L \circ (\psi_{\beta\alpha}^{E_1}(p))^{-1} \right)$$

To see this note that:

If $u = \psi_{\beta\alpha}^{E_1}(p)(v)$ and $w = L(v)$ then

$$\psi_{\beta\alpha}^{\hom(E_1,E_2)}(p)(L)\left(\psi_{\beta\alpha}^{E_1}(p)(v)\right) = \psi_{\beta\alpha}^{E_2}(p)(L(v)).$$



<u>Tensor Products:</u> $E_i \longrightarrow M$ $i = 1, 2$ vector bundles

The tensor product bundle

$E_1 \otimes_{\mathbb{R}} E_2 \longrightarrow M$ is given by the transition functions $\psi^1_{\beta\alpha} \otimes \psi^2_{\beta\alpha}$.

Ex $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_{2\times 2}$ $B = \begin{bmatrix} b_{ij} \end{bmatrix}_{3\times 3}$

$A \otimes B : \mathbb{R}^2 \otimes \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \otimes \mathbb{R}^3$ $\quad \mathbb{R}^2 : e_i$

$A \otimes B = \begin{bmatrix} a_1 B & a_2 B \\ a_3 B & a_4 B \end{bmatrix}_{6\times 6}$ $\quad \mathbb{R}^3 : f_j$ $\quad \mathbb{R}^2 \otimes \mathbb{R}^3 : e_i \otimes f_j$

$(A \otimes B)(e_i \otimes f_j) \doteq (Ae_i) \otimes (Bf_j)$

Special Case: $k_1 = k_2 = 1$.

$E_i \longrightarrow M$ line bundles. Then $E_1 \otimes E_2 \longrightarrow M$ is also a line bundle and its transition function $\psi^1_{\alpha\beta} \cdot \psi^2_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbb{R}^* \,(\text{or } \mathbb{C}^*)$

Ex $\mathcal{O}(k) \longrightarrow \mathbb{CP}^1$, $\psi_{01}(z) = [1/z^k]$

$\mathcal{O}(k_1) \otimes \mathcal{O}(k_2) \longrightarrow \mathbb{CP}^1$ is a line bundle with transition function $z \longmapsto [1/z^{k_1} \cdot 1/z^{k_2}]$ and thus $\mathcal{O}(k_1) \otimes_{\mathbb{C}} \mathcal{O}(k_2) = \mathcal{O}(k_1 + k_2)$.

Note that $\mathcal{O}(k_1) \otimes \mathcal{O}(k_1) = \mathcal{O}(0)$ is the trivial vector bundle $\mathcal{O}(0) \simeq \mathbb{CP}^1 \times \mathbb{C}$

<u>Dual of $\mathcal{O}(k)$</u>: $\mathcal{O}(k)^* = \hom(\mathcal{O}(k), \mathcal{O}(0))$
$$= \mathcal{O}(-k).$$

<u>Pull back of a vector bundle:</u>

$N \xrightarrow{f} M$, $\quad P : E \longrightarrow M$

$f^*(E) \longrightarrow N$

$$\begin{array}{ccccc} f^*(E) & \dashrightarrow & E & \ni & \underline{v} \\ {\scriptstyle P'}\downarrow & & \downarrow {\scriptstyle P} & & \underline{P}(v) \\ \underline{p \in N} & \xrightarrow[f]{} & M & & \| \\ & & & & f(p) \end{array}$$

$f^*(E) = \{(p, v) \in N \times E \mid f(p) = P(v)\}$

$P'(p) = P^{-1}(f(p)) = E_{f(p)}$

In terms of transition functions: If $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(k, \mathbb{R})$ are the transition functions for $E \longrightarrow M$ then

$\psi_{\alpha\beta} \circ f : f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) \longrightarrow GL(k, \mathbb{R})$

are the transition functions for $f^*(E) \to N$.

De Rham Cohomology (Chapter 4)

M smooth manifold of dimension n.

Chain complex:

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0$$

$$\dots \to \Omega^i(M) \xrightarrow{d_i} \Omega^{i+1}(M) \xrightarrow{d_{i+1}} \Omega^{i+2}(M) \to \dots$$

We know that $d_{i+1} \circ d_i = 0, \forall i$.

Hence $\mathrm{Im}(d_i) \subseteq \ker(d_{i+1}), \forall i$.

$$H^{i+1}_{DR}(M) = \frac{\ker(d_{i+1})}{\mathrm{Im}(d_i)} \begin{array}{l} \leftarrow \text{closed } i+1\text{-forms} \\ \leftarrow \text{exact } i+1\text{-forms} \end{array}$$

Remark: If M is a compact manifold then de Rham cohomology groups are all finite dimensional.

Example: If M is a smooth manifold, then $H^0_{DR}(M) \simeq \mathbb{R}^{b_0}$, where b_0 is the number of connected components of M. In particular, if M is connected then $H^0_{DR}(M) \simeq \mathbb{R}$.

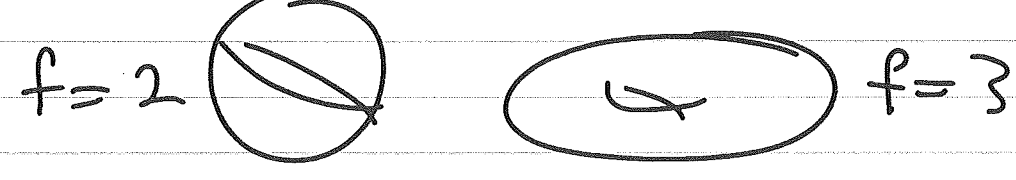
Prof $H^0_{DR}(M) = \dfrac{\text{Closed 0-forms} = \mathbb{R}^{b_0}}{\text{Exact 0-forms} = (0)} = \mathbb{R}^{b_0}$

$\Omega^{-1}(M) \xrightarrow{d} \Omega^0(M) \Rightarrow$ There is no exact zero forms.
$\Omega^{-1}(M) = \emptyset$

<u>Closed 0-forms:</u> $f: M \to \mathbb{R}$ smooth function so that $df = 0$ on M. In a local coordinate chart $(x_1, \ldots, x_n)$ then $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$.

So, $\frac{\partial f}{\partial x_i} = 0$ on M.

Hence f is locally constant on M.



So the vector space of closed 0-forms is the vector space of locally constant functions on M. Hence H is isomorphic to $\mathbb{R}^{b_0}$.

<u>Proposition:</u> If $f: M \to N$ is a smooth map then there is a vector space homomorphism $f^*: H^k_{DR}(N) \longrightarrow H^k_{DR}(M)$ defined by

$f^*([\omega]) \doteq [f^*\omega]$.

Proof: Since $d \circ f^* = f^* \circ d$ we see that if ω is closed then $d(f^*(\omega)) = f^*(d\omega) = 0$ (with $d\omega = 0$) so that $f^*(\omega)$ is also closed.

It is well-defined: If $[\omega_1] = [\omega_2]$ then $[f^*(\omega_1)] = [f^*(\omega_2)]$.

Proof: $[\omega_1] = [\omega_2] \Rightarrow \omega_1 - \omega_2$ is exact.

$\Rightarrow \omega_1 - \omega_2 = d\eta$, for some $\eta \in \Omega^{k-1}(N)$ ($\omega_i \in \Omega^k(N)$). Then

$$f^*(\omega_1) - f^*(\omega_2) = f^*(\omega_1 - \omega_2) = f^*(d\eta) = d(f^*(\eta))$$

$\Rightarrow [f^*(\omega_1)] = [f^*(\omega_2)]$.

Remark: $H^*_{DR}(M) = \bigoplus_{k=0}^{n} H^k_{DR}(M)$

$[\omega] \in H^k_{DR}(M)$, $[\eta] \in H^l_{DR}(M)$. Then $\omega \wedge \eta$ is a $k+l$-form. Moreover,

$$d(\omega \wedge \eta) = \underset{\overset{\parallel}{0}}{d\omega} \wedge \eta + (-1)^k \omega \wedge \underset{\overset{\parallel}{0}}{d\eta} = 0.$$

$\Rightarrow \omega \wedge \eta$ is closed. So we can define

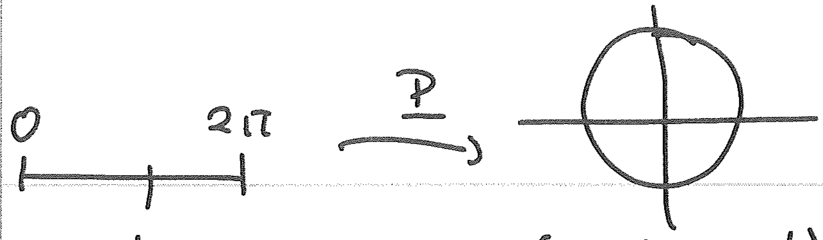
$[\omega][\eta] \doteq [\omega \wedge \eta]$ is a product on $H^*_{DR}(M)$.

__Exercise:__ This product is well defined.

__Proposition:__ If $f: M \to N$ is a smooth map then $f^*: H^*_{DR}(N) \to H^*_{DR}(M)$ is an $\mathbb{R}$-algebra homomorphism, preserving the degrees.

__Proposition:__ The map $I: H^1_{DR}(S^1) \to \mathbb{R}$ given by $I([\omega]) = \int_{S^1} \omega$ is an $\mathbb{R}$-vector space isomorphism.

__Proof:__ Consider the 1-form $\omega = \frac{1}{2\pi}(x\,dy - y\,dx)$ on $\mathbb{R}^2$. Since S^1 is 1-dimensional $d\omega = 0$ on S^1. Hence, ω is closed.



$t \longmapsto (\cos t, \sin t)$ $\quad \omega = \frac{1}{2\pi}(x\,dy - y\,dx)$

$\quad\quad\quad\quad\quad\quad\; \overset{\shortparallel}{x} \quad \overset{\shortparallel}{y}$

$$P^*(\omega) = \frac{1}{2\pi}\left(\cos t \, d(\sin t) - \sin t \, d(\cos t)\right)$$
$$= \frac{1}{2\pi}\left(\cos^2 t \, dt + \sin^2 t \, dt\right) = \frac{dt}{2\pi}$$

Hence $\int_{S^1} \omega = \int_0^{2\pi} P^*(\omega) = \int_0^{2\pi} \frac{dt}{2\pi} = \underline{1}.$

So the map $I : H^1_{DR}(S^1) \longrightarrow \mathbb{R}$ is an onto $\mathbb{R}$-linear map: $I(a[\omega] + b[\eta]) = a\, I([\omega]) + b\, I([\eta])$.

To finish the proof we must show $\ker(I) = (0)$.

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Note Title 4.03.2020

Claim: $\mathcal{I} : H^1_{DR}(S^1) \longrightarrow \mathbb{R}$ is injective.

$[\omega] \longmapsto \int_{S^1} \omega$

Proof. Assume that $\mathcal{I}([\nu]) = 0 \Rightarrow \int_{S^1} \nu = 0$.

must show: $[\nu] = 0 \Leftrightarrow \nu$ is exact.

$P : \mathbb{R} \longrightarrow S^1$, $P(t) = (\cos t, \sin t)$, $t \in \mathbb{R}$.

$0 = \int_{S^1} \nu = \int_0^{2\pi} \underbrace{P^* \nu}$ $\qquad P^*\nu$ is a 1-form on $\mathbb{R}$.

So $P^*\nu = f(t)\,dt$, $f : \mathbb{R} \to \mathbb{R}$

$0 = \int_0^{2\pi} f(t)\,dt$. Let $F(t) = \int_0^t f(s)\,ds$.

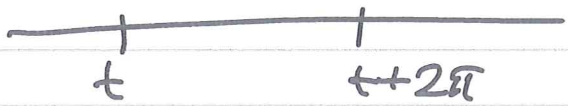
Note that $F(t+2\pi) - F(t) = \int_t^{t+2\pi} f(s)\,ds = \int_0^{2\pi} f(s)\,ds$

$= 0$.

$\nu(P(t)) = \nu(\underbrace{P(t+2\pi)}_{q(t)}) \Rightarrow P^*\nu = q^*\nu$

$f(t)\,dt = f(t+2\pi)\,dt \quad \Leftarrow$

So $F(t)$ is periodic with period 2π.



$F(t) = F(t+2\pi)$

$\Rightarrow$ $$\begin{array}{ccc} \mathbb{R} & \xrightarrow{F} & \mathbb{R} \\ P \downarrow & \nearrow \tilde{F} & \\ S^1 & & \end{array}$$

So there is a unique smooth function $\tilde{F}: S^1 \to \mathbb{R}$ so that

$$\tilde{F} \circ P = F.$$

$$F(t+2\pi) = \int_0^{t+2\pi} f(s)\,ds = \int_0^t f(s)\,ds + \int_t^{t+2\pi} f(s)\,ds$$

$$= F(t) + \underbrace{\int_0^{2\pi} f(s)\,ds}_{=0}$$

$$= F(t)$$

Now, $\underline{P^*(d\tilde{F})} = d(P^*(\tilde{F}))$

$\tilde{F}: S^1 \to \mathbb{R}$ $\quad = d(\tilde{F} \circ P)$

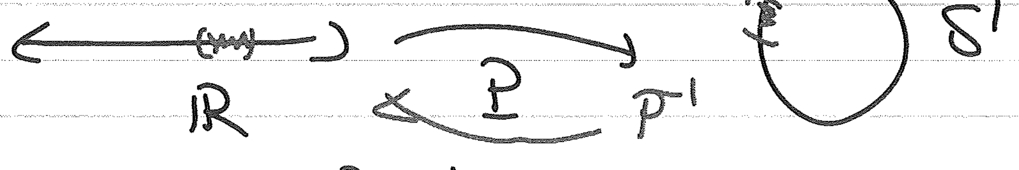
$$= dF$$

$$= \underline{P^* \nu}$$

because $F(t) = \int_0^t f(s)\,ds \Rightarrow dF = f(t)\,dt$

$$= P^*(\nu)$$

$\Rightarrow$ $\underline{P}^*(\underline{\nu - d\tilde{F}}) = 0$



$P(t) = (\cos t, \sin t)$ is local diffeomorphism

Hence, $\nu - d\tilde{F} = 0 \Rightarrow \nu = d\tilde{F}$

$\Rightarrow [\nu] = 0.$

This finishes the proof. ∎

<u>Proposition:</u> M compact smooth manifold without boundary. Assume that M is orientable. Let $\omega \in \Omega^n(M)$ be an exact n-form. Then $\int_M \omega = 0$. Thus the map $H^n_{DR}(M) \longrightarrow \mathbb{R}$

$[\omega] \longmapsto \int_M \omega$

is an $\mathbb{R}$-linear homomorphism.

<u>Proof:</u> Well-definedness: Let $[\omega_1] = [\omega_2]$ in $H^n_{DR}(M)$. Then $\omega_1 - \omega_2 = d\nu$, for some $n-1$-form $\nu \in \Omega^{n-1}(M)$. Thus

$$\int_M \omega_1 = \int_M \omega_2 + d\nu = \int_M \omega_2 + \int_M d\nu$$

$$= \int_M \omega_2 + \underbrace{\int_{\partial M = \emptyset} \nu}_{=0}$$

$$= \int_M \omega_2.$$

Linearity is obvious.

Summary $H^k_{DR}(S^1) = \begin{cases} \mathbb{R} & k=0,1. \\ 0 & k \geq 2. \end{cases}$

Definition: X topological space and $A \subseteq X$ subspace. A continuous map $r: X \to A$ is called a retraction if $r \circ i : A \to A$ is identity, where $i : A \hookrightarrow X$ is the inclusion map.

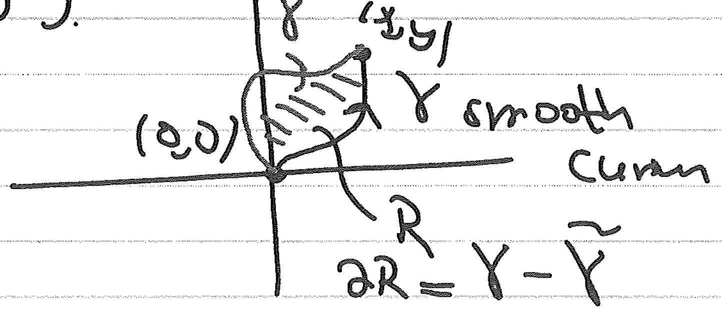
Theorem: There is no smooth retraction $r: D^2 \to \partial D^2 = S^1$.

Before the proof we need some preparation.

Proposition: $H^1_{DR}(\mathbb{R}^2) = H^1_{DR}(D^2) = 0$.

Proof. Let $\omega = f(x,y)\,dx + g(x,y)\,dy$ be a closed 1-form on $\mathbb{R}^2$ on D^2. Since ω is closed $0 = d\omega = (g_x - f_y)\,dx \wedge dy$ and thus $g_x = f_y$ on $\mathbb{R}^2$ (D^2).

Aim: To show that ω is exact!

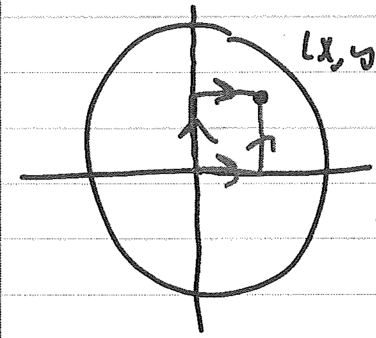


Define $F(x,y) \doteq \int_\gamma \omega$, $F : \mathbb{R}^2 \to \mathbb{R}$ smooth function.

$$\int_\gamma \omega - \int_{\tilde\gamma} \omega = \int_{\gamma - \tilde\gamma} \omega = \int_{\partial R} \omega = \int_R d\omega = 0.$$

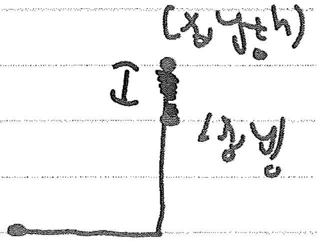
$\Rightarrow \int_\gamma \omega = \int_{\tilde\gamma} \omega \Rightarrow F$ is well defined.

<u>Claim:</u> $dF = F_x\,dx + F_y\,dy = \omega$



$$F(x,y) = \int f(x,y)\,dx + g(x,y)\,dy$$

<u>must show:</u> $F_x = f$, $F_y = g$



$$F_y = \lim_{h \to 0} \frac{F(x_0, y_0 + h) - F(x_0, y_0)}{h}$$

$$= \lim_{h \to 0} \frac{\int_I f(x_0,y_0)\,dx + g(x_0,y_0)\,dy}{h}$$

$x = x_0$
$y = y_0 + h$
$dy = dh$
$dx = 0$

$$= \lim_{h \to 0} \frac{\int_{y_0}^{y_0+h} g(x_0, y_0 + h)\,dh}{h}$$

$= g(x_0, y_0) \Rightarrow F_y = g.$

Similarly, $F_x = f$ and thus $\omega = dF$. ∎

Proof of the Theorem: Assume on the contrary that there is a smooth retraction $r: D^2 \longrightarrow \partial D^2 = S^1$. Since $r \circ i = id_{S^1}$ we see that the composition

$$\underset{\mathbb{R}}{\overset{}{}} \; H^1_{DR}(S^1) \xrightarrow{r^*} H^1_{DR}(D^2) \xrightarrow{i^*} H^1_{DR}(S^1)$$

with $H^1_{DR}(S^1) \cong \mathbb{R}$, $H^1_{DR}(D^2) = 0$, $H^1_{DR}(S^1) \cong \mathbb{R}$,

is also identity. $Id = (r \circ i)^* = i^* \circ r^*$

This is a contradiction since $r^* = 0$, because $H^1_{DR}(D^2) = 0$. Hence, there is no such retraction.

Poincaré Lemma: Let $I \subseteq \mathbb{R}$ be an interval. Then for any smooth manifold M we have

$$H^k_{DR}(M \times I) \cong H^k_{DR}(M).$$

Proof: $Pr: M \times I \longrightarrow M$, $Pr(x,t) = x$. Let $a \in I$ and consider the inclusion map $i_a: M \longrightarrow M \times I$ given by $i_a(x) = (x, a)$, $x \in M$. Note that $Pr \circ i_a: M \longrightarrow M$, $x \longmapsto (x,a) \longmapsto x$, is the identity map of M. Hence, the composition

$$H^k_{DR}(M) \xrightarrow{pr^*} H^k_{DR}(M\times I) \xrightarrow{i_a^*} H^k_{DR}(M) \text{ is}$$

identity. $(pr\circ i_a)^* = id$

Let $U\subseteq M$ be a coordinate chart. Locally $M\times I$ can be seen as $U\times I$. Let $x_1,\dots,x_n$ be the coordinates on U.

$I=(i_1,\dots,i_{k-1})$, $J=(j_1,\dots,j_k)$ and consider forms of type $f(x,t)\, dx_I \wedge dt$ and $g(x,t)\, dx_J$.

$dx_I = dx_{i_1}\wedge\dots\wedge dx_{i_{k-1}}$

Consider the map

$$P\left(f(x,t)\, dx_I\wedge dt\right) \doteq (-1)^{k-1}\left(\int_0^t f(x,s)\,ds\right) dx_I$$

and $P\left(g(x,t)\,dx_J\right) \doteq 0$.

$$P: \Omega^k(M\times I) \longrightarrow \Omega^{k-1}(M).$$

<u>Claim:</u> 1) $(d\circ P + P\circ d)(f(x,t)\,dx_I\wedge dt) = f(x,t)\,dx_I\wedge dt$

and 2) $(d\circ P + P\circ d)(g(x,t)\,dx_J) = (g(x,t)-g(x,a))\,dx_J$.

It follows that for the composition

$i_a\circ pr: M\times I \longrightarrow M\times I$, we have

$(pr^* \circ i_a^*)(f(x,t)\, dx_I \wedge dt) = 0$, and

$M \to M \times I$
$\quad\quad\quad\quad\quad t$

$(pr^* \circ i_a^*)(g(x,t)\, dx_J) = g(x,a)\, dx_J$.

So, for any $\omega \in \Omega^k(M \times I)$, we have

$(d \circ P + P \circ d)(\omega) = \omega - (pr^* \circ i_a^*)(\omega)$.

Hence, $[\omega] - [(i_a \circ pr)^*(\omega)] = [d(\underbrace{P(\omega)})] + [\underbrace{P\, d(\omega)}]$ (provided ω is closed)

(exact $\Rightarrow$ " 0; second term " 0)

$\Rightarrow [\omega] = (i_a \circ pr)^* [\omega]$.

$\Rightarrow (i_a \circ pr)^*$ is also identity.

$Id = pr^* \circ i_a^* \Rightarrow i_a^*$ is injective and pr^* is surjective.

$Id = i_a^* \circ pr^* \Rightarrow i_a^*$ is surjective and pr^* is injective.

$\Rightarrow$ Both i_a^* and pr^* are isomorphisms. ∎

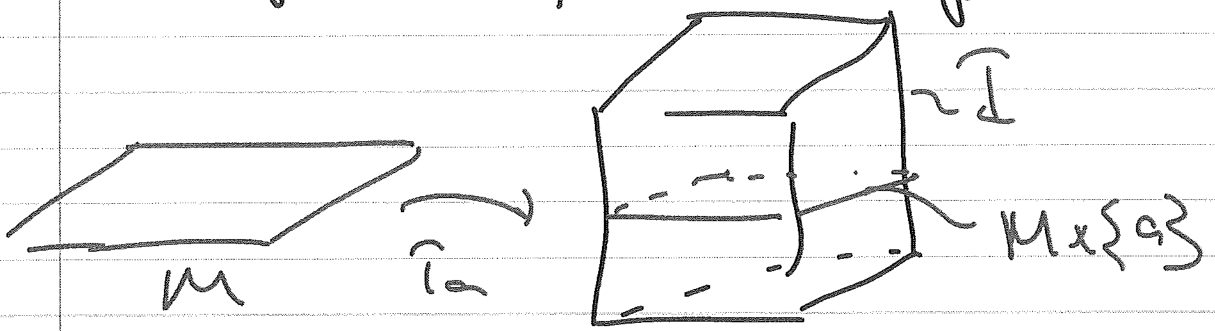
<u>Corollary</u> For any smooth manifold M and integer $k \geq 0$ we have

$$H^i_{DR}(M \times \mathbb{R}^k) \cong H^i_{DR}(M), \text{ for any } i \geq 0.$$

In particular, for $i > 0$,

$H^i_{DR}(\mathbb{R}^k) = H^i_{DR}(\{pt\} \times \mathbb{R}^k) \cong H^i_{DR}(\{pt\}) = 0.$

Moreover, using similar considerations the homomorphism $i_a^* : H^k_{DR}(M \times I) \to H^k_{DR}(M)$ is independent of the choice of $a \in I$.



Last time: $I \subseteq \mathbb{R}$, $a \in I$, $i_a : M \longrightarrow M \times I$, $x \longmapsto (x, a)$

inclusion map. Then

$i_a^* : H^k_{DR}(M \times I) \longrightarrow H^k_{DR}(M)$ is an isomorphism and the map i_a^* is independent of the choice of $a \in I$.

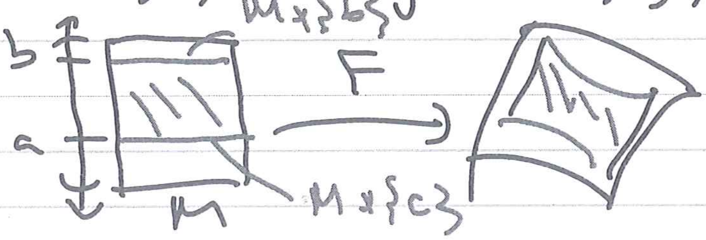
<u>Remark:</u> The proof of independence from the point $a \in I$ is done by showing

$$\left.\frac{d}{dt}\right|_{t=a} (i_t^* \omega) = d(\cdots) \Rightarrow \frac{d}{dt}[i_t^* \omega] = 0.$$

$\Rightarrow [i_t^* \omega]$ is independent of t.

Hence $[i_a^* \omega] = [i_b^* \omega]$, for all $a, b \in I$.

A smooth map $F : M \times I \to N$, where I is an interval is called a smooth homotopy from $f(x) = F(x, a)$ to $g(x) = F(x, b)$, for any $a, b \in I$.

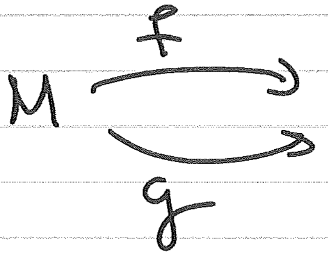


Note that $f(x) = f(x,a) = F \circ i_a$ and

$g(x) = F(x,b) = F \circ i_b$. Then

$f^* = (F \circ i_a)^* = i_a^* \circ F^* = i_b^* \circ F^* = (F \circ i_b)^* = g^*$

as maps $f^* = g^* : H^k_{DR}(N) \to H^k_{DR}(M)$



$f^*([\omega]) = g^*([\omega])$

for any $[\omega] \in H^k_{DR}(N)$.

<u>Definition</u>: Two functions $f: X \to Y$ and

$g: Y \to X$ (continuous) of topological

spaces are called homotopy equivalence if

$g \circ f$ is homotopic to id_X and

$f \circ g$ is homotopic to id_Y.

[$\phi_1 : X \to Y$, $\phi_2 : X \to Y$ are called

homotopic if there is a continuous

map $\Phi : X \times [a,b] \to Y$ so that

$\Phi(x,a) = \phi_1(x)$ and $\Phi(x,b) = \phi_2(x)$ for

all $x \in X$.]

If $f: M \to N$ and $g: N \to M$ are smooth homotopy inverses of each other, i.e.

$g \circ f: M \to M \Rightarrow g \circ f \simeq id_M$

$f \circ g: N \to N \Rightarrow f \circ g \simeq id_N$.),

then we get

$$H^k_{DR}(M) \xrightarrow{g^*} H^k_{DR}(N) \xrightarrow{f^*} H^k_{DR}(M)$$

$$(id_M)^* = Id_{H^k_{DR}(M)}$$

$f^* \circ g^* = Id_{H^k_{DR}(M)}$.

Similarly $g^* \circ f^* = Id_{H^k_{DR}(N)}$, so that both f^* and g^* are isomorphisms.

<u>Corollary</u>: $\mathbb{R}^n$ and D^n has trivial cohomologies.

<u>Proof</u>: $f: D^n \to \{0\}$, $g: \{0\} \to D^n$, $0 \mapsto 0$.

$f \circ g: \{0\} \to \{0\}$ and thus $f \circ g = id_{\{0\}}$.

Also, $g \circ f: D^n \to D^n$, $(g \circ f)(x) = 0 \ \forall x$.

Let $\psi: D^n \times [0,1] \to D^n$; $\psi(x,t) = tx$, $t \in [0,1]$, $x \in D^n$.

$\Psi(x,0) = 0 = (g\circ f)(x)$ ⟵

$\Psi(x,1) = x = \mathrm{id}_{D^n}$ so that $g\circ f \simeq \mathrm{id}_{D^n}$.

Hence $f^*: H^k_{DR}(\{0\}) \longrightarrow H^k_{DR}(D^n)$

must be an isomorphism. However,

$$H^k_{DR}(\{0\}) = \begin{cases} \mathbb{R} & \text{if } k=0 \\ 0 & \text{if } k>0 \end{cases}$$

$$\Rightarrow H^k_{DR}(D^n) = \begin{cases} \mathbb{R} & \text{if } k=0 \\ 0 & \text{if } k>0. \end{cases}$$

Theorem The homomorphism

$I : H^2_{DR}(S^2) \longrightarrow \mathbb{R}$, $[\omega] \mapsto \int_{S^2} \omega$, is

an isomorphism of vector spaces.

Proof: $\omega_0 = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$

We know that $\int_{S^2} \omega_0 = 4\pi$ if S^2 is the unit sphere.

Hence, I is onto.

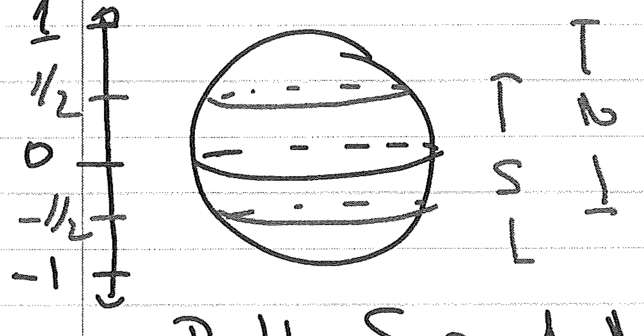
must show: $\ker I = (0)$.

In other words, if $\underline{I(\omega) = 0}$ then

$\underline{\omega = d\nu}$ for some 1-form ν.

Define subset $N = \{(x,y,z) \in S^2 \mid z > -1/2\}$

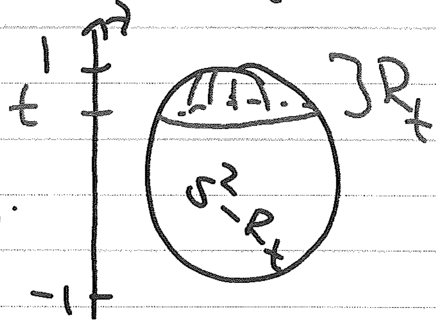
$S = \{(x,y,z) \in S^2 \mid z < 1/2\}$.



$S \cap N = \{(x,y,z) \mid -1/2 < z < 1/2\}$.

Both S and N are diffeomorphic to D^2 and thus $H^2_{DR}(S) = H^2_{DR}(N) = H^2_{DR}(D^2) = (0)$.

Let $I(\omega) = 0$ for some 2-form ω on S^2.
$\omega|_N$ restriction of ω to N. Since
$[\omega|_N] \in H^2_{DR}(N) = (0)$ we get
$\omega|_N = d\nu_N$ for some 1-form ν_N on N.
Similarly, $\omega|_S = d\nu_S$ for some 1-form
ν_S on S.
Let $R_t = \{(x,y,z) \in S^2 \mid z > t\}$.



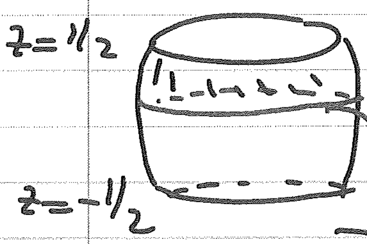
$$0 = \int_{S^2} \omega = \int_{\overline{R_t}} \omega + \int_{S^2 \setminus R_t} \omega = \int_{\overline{R_t}} d\nu_N + \int_{S^2 \setminus R_t} d\nu_S.$$

$t \in (-1/2, 1/2)$

$$\Rightarrow \int_{\overline{R_t}} d\nu_N = -\int_{S^2 \setminus R_t} d\nu_S$$

$$\int_{\partial \overline{R_t}} \nu_N = -\int_{\partial(S^2 \setminus R_t) = -\partial \overline{R_t}} \nu_S = \int_{\partial \overline{R_t}} \nu_S$$

$\Rightarrow \int_{\partial \overline{R}_t} \nu_N - \nu_S = 0$, for all $t \in (-1/2, 1/2)$.



Clearly $\partial \overline{R}_t$ is homotopy equivalent to $\overline{S \cap N}$.

$H^1_{DR}(S \cap N) \simeq H^1_{DR}(\partial \overline{R}_t)$ and $\nu_N - \nu_S$ is exact on $\partial \overline{R}_t$. $\Rightarrow [\nu_N - \nu_S] = 0$ in $H^1_{DR}(\partial \overline{R}_t) = H^1_{DR}(S \cap N)$.

Hence, $\nu_N - \nu_S = df$, for some smooth function $f : S \cap N \to \mathbb{R}$. Extend f to S, call it f again $f : S \to \mathbb{R}$. Consider the 1-form on S^2 ($S^2 = S \cup N$) defined by

$$\nu(p) = \begin{cases} \nu_N(p), & p \in N \\ \nu_S(p) + df(p), & p \in S. \end{cases}$$

If $p \in N \cap S$, $\nu_S(p) + df(p) = \cancel{\nu_S(p)} + \nu_N(p) - \cancel{\nu_S(p)}$
$= \nu_N(p)$.

On the other hand,

$$d\nu(p) = \begin{cases} d\nu_N(p), & p \in N \\ d\nu_S(p) + \cancel{d^2 f(p)}, & p \in S \end{cases}$$

(with $d^2 f(p) = 0$)

$$d\nu(p) = \begin{cases} \omega|_N(p), & p \in N \\ \omega|_S(p), & p \in S \end{cases} = \omega(p).$$

So $[\omega] = [d\nu] = 0$ in $H^2_{DR}(S^2)$.

Some Applications

Winding number: $\mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow S^1 \times \mathbb{R}$

$$p \longmapsto \left(\frac{p}{\|p\|}, \ln\|p\|\right)$$

is a diffeomorphism.

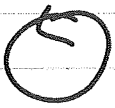
Hence $H^1_{DR}(\mathbb{R}^2 \setminus \{(0,0)\}) \cong H^1_{DR}(S^1 \times \mathbb{R})$

$\cong H^1_{DR}(S^1)$ (Poincaré Lemma)

$\cong \mathbb{R}$

Let $\omega = \frac{1}{2\pi} \frac{x\,dy - y\,dx}{x^2+y^2} \in \Omega^1(\mathbb{R}^2 \setminus \{(0,0)\})$.

$\int_{S^1} \omega = 1 \neq 0$ $\quad x = \cos t$, $y = \sin t$, $t \in [0, 2\pi]$

$S^1 =$ 

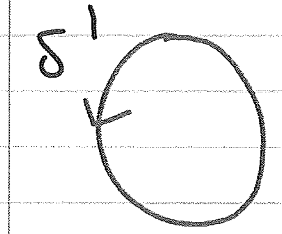
So $[\omega]$ is closed but not exact.

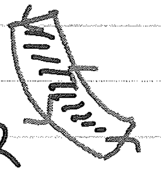
$H^1_{DR}(\mathbb{R}^2 \setminus \{(0,0)\}) \cong \langle [\omega] \rangle \cong \mathbb{R}$.

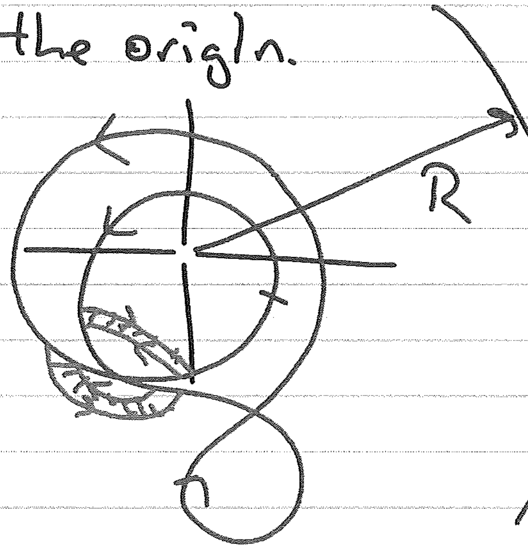
Winding Number: Let $f : S^1 \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$ smooth function. Then the Winding number of f is the integral $\int_{S^1} f^* \omega$.

This number is always an integer and

It is nothing but the number of times $f(S^1)$ winds around the origin.

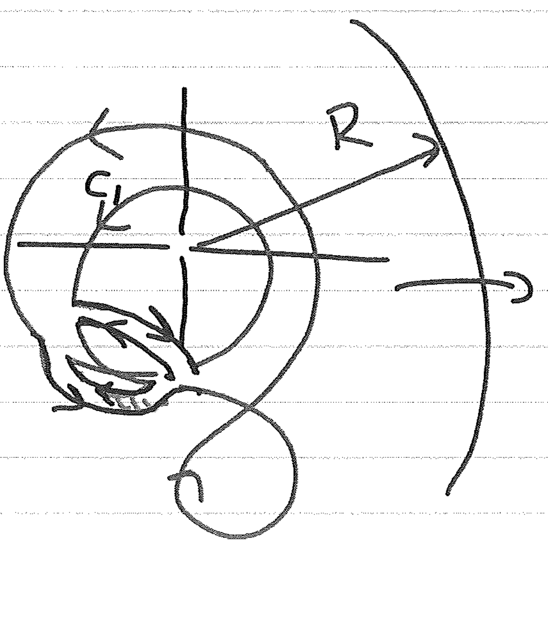


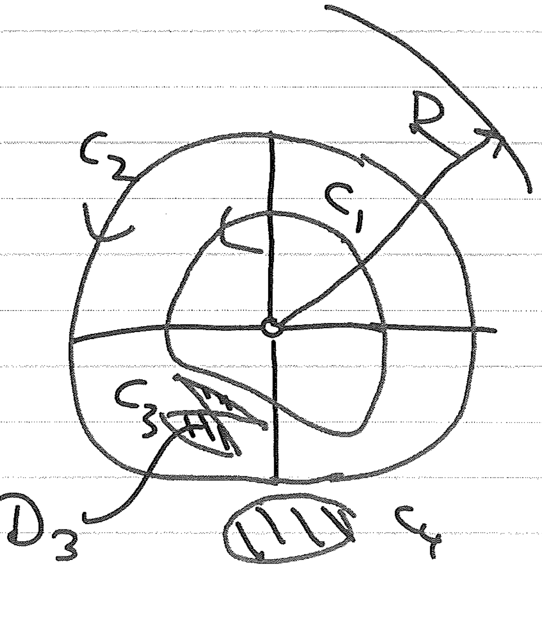




$$0 = \int_R \underset{0}{\underset{\shortparallel}{d\omega}} = \int_{\partial R} \omega = \int \omega + \int \omega$$

$$\Rightarrow \int \omega = \int \omega$$

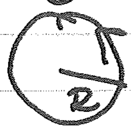


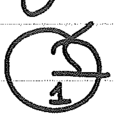


$$\int_{S^1} f^*\omega = \int_{C_1} \cdots + \int_{C_2} \cdots + \underbrace{\int_{C_3} \cdots}_{=0} + \underbrace{\int_{C_4}}_{=0 \text{ by Stokes}} \qquad C_3 = \partial D_3$$

$$\int_{C_3} f^*\omega = \int_{\partial D_3} f^*\omega = \int_{D_3} d f^*\omega = \int_{D_3} f^* \underbrace{d\omega}_{=0} = 0$$

$$\int_{C_1} f^*\omega = \int f^*\omega = \int_{S^1} f^*\omega = 1.$$

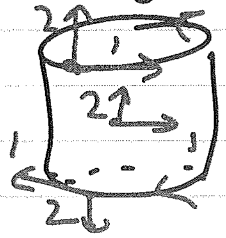




Similarly, $\int_{C_2} f^*\omega = 1$.

<u>Theorem</u>: If $F : S^1 \times [0,1] \to \mathbb{R}^2 \setminus \{(0,0)\}$ is a smooth homotopy from $f(p) = F(p,0)$ to $g(p) = F(p,1)$, then $W(f) = W(g)$.

<u>Proof</u>:



$S^1 \times I$

$$0 = \int_{S^1 \times I} F^*(d\omega) = \int_{S^1 \times I} d(F^*\omega) = \int_{\partial(S^1 \times I)} F^*\omega$$

($d\omega = 0$)

$$\partial(S^1 \times I) = S^1 \times \{1\} - S^1 \times \{0\}$$

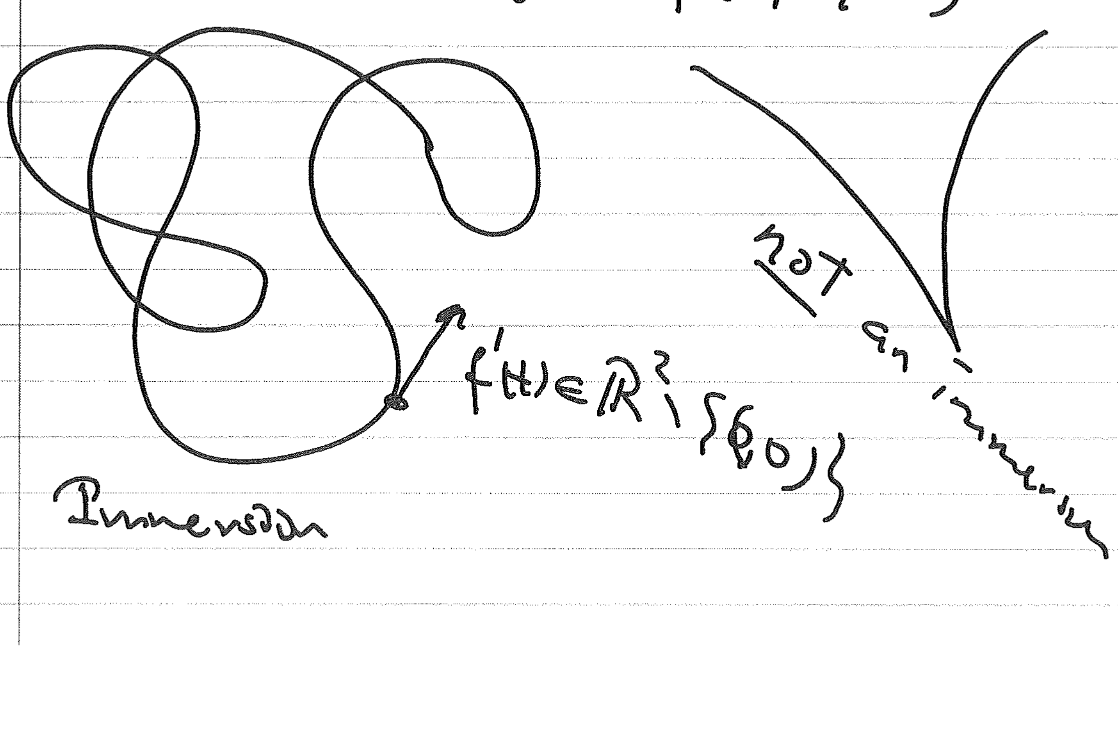
$$0 = \int_{S^1 \times \{1\}} F^*\omega - \int_{S^1 \times \{0\}} F^*\omega \qquad F|_{S^1 \times \{1\}} = g$$

$$F|_{S^1 \times \{0\}} = f$$

$$= \int_{S^1 \times \{1\}} g^*\omega - \int_{S^1 \times \{0\}} f^*\omega$$

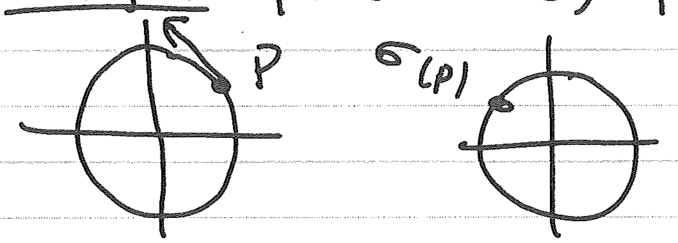
$$\Rightarrow W(f) = W(g). \quad \blacksquare$$

Rotation Number: Let $f : S^1 \to \mathbb{R}^2$ be an immersion (i.e. $f'(t) \neq 0$).



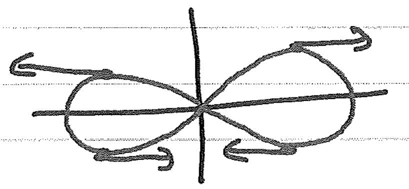
Definition: The Rotation number of an immersion $f: S^1 \to \mathbb{R}^2$ is defined to be the winding number of $\sigma: S^1 \to \mathbb{R}^2 \setminus \{(0,0)\}$ defined by $\sigma(p) = f'(p)$.

Example $f: S^1 \to \mathbb{R}^2$, $f(p) = p$.



$\mathrm{Rot}(f) = 1.$

Ex



$\mathrm{Rot}(f) = 0.$

$f(\theta) = (\sin\theta, \sin\theta\cos\theta), \quad \theta \in [0, 2\pi].$

$f'(\theta) = (\underset{x}{\cos\theta}, \underset{y}{\cos 2\theta}), \quad \omega = \frac{1}{2\pi}\frac{x\,dy - y\,dx}{x^2+y^2}$

$f^*\omega = \frac{\cos 2\theta \sin\theta - 2\cos\theta \sin 2\theta}{2\pi(\cos^2\theta + \cos^2 2\theta)}$ ← odd function

$$\mathrm{Rot}(f) = \int_0^{2\pi} f^*\omega = \int_{-\pi}^{\pi} f^*\omega = 0.$$

Theorem If $f: S^1 \to \mathbb{R}^2$, $g: S^1 \to \mathbb{R}^2$ are two immersions, which are *homotopic through immersions* then $Rot(f) = Rot(g)$.

Proof: $F: S^1 \times [0,1] \longrightarrow \mathbb{R}^2$

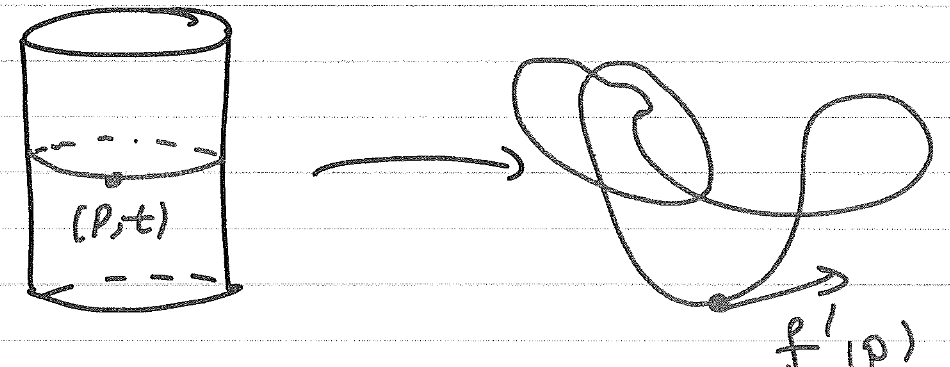
$F(p,0) = f(p)$, $F(p,1) = g(p)$ and

$F|_{S^1 \times \{1\}} : S^1 \times \{1\} \longrightarrow \mathbb{R}^2$ is an immersion.

In this case $\sigma_f = f'$ and $\sigma_g = g'$ are homotopic via G, when

$G: S^1 \times I \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$

$G(p,t) = f_t'(p)$, when $f_t(p) = F(p,t)$.



In particular, σ_f and σ_g are homotopic. Here $Rot(f) = W(\sigma_f) = W(\sigma_g) = Rot(g)$. ∎

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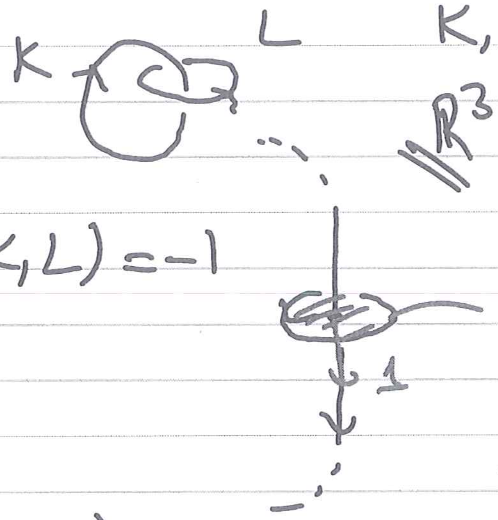
Linking Number:

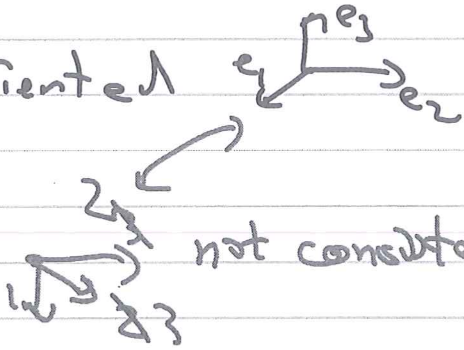
K, L two disjoint knots in $\mathbb{R}^3$.

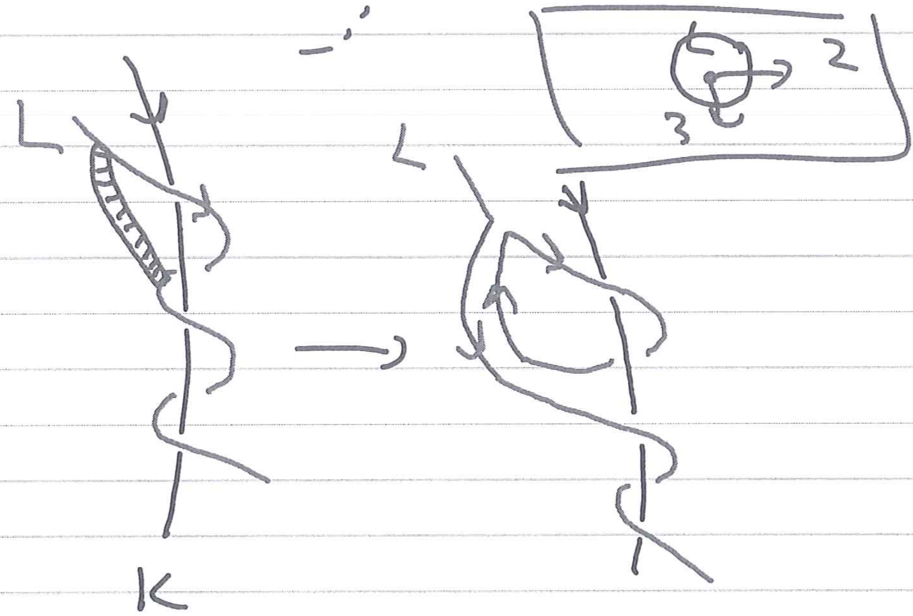
K, L both oriented

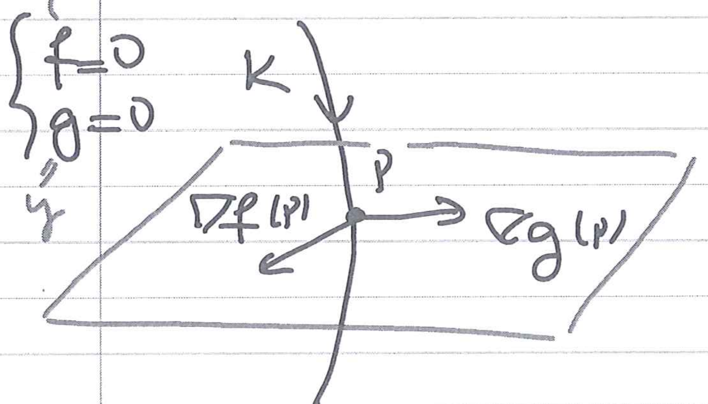
$\mathbb{R}^3$ oriented



$\ell(K,L) = -1$







$$\omega_K = \frac{1}{2\pi}\frac{x\,dy - y\,dx}{x^2+y^2}$$

$$= \frac{1}{2\pi}\frac{f\,dg - g\,df}{f^2+g^2}$$

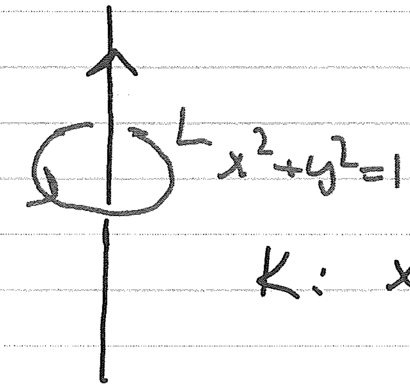
ω_K is called the Linking form of the knot K.

$$\ell(K,L) = \int_L \omega_K = \pm \int_K \omega_L$$

<u>Example</u>: $K, L \subseteq S^3$, K: z-axis $\cup \{\infty\}$

$L: x^2+y^2=1,\ z=0$

$S^3 = \mathbb{R}^3 \cup \{\infty\}$



$K: \ x=0,\ y=0,\ \omega_K = \frac{1}{2\pi} \frac{x\,dy - y\,dx}{x^2+y^2}$

$$\int_L \omega_K = 1. \qquad \int_K \omega_L = ?$$

$L: \ x^2+y^2-1=0$
$\quad z=0$

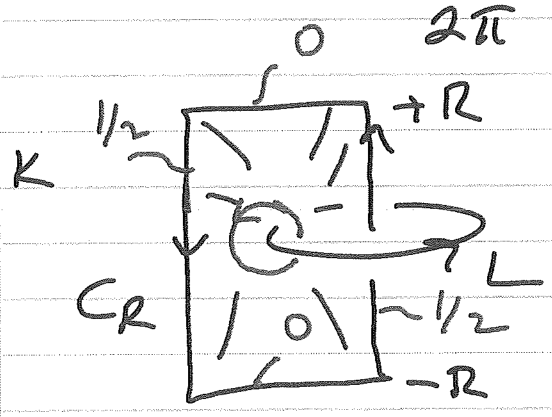
$$\omega_L = \frac{1}{2\pi} \frac{(x^2+y^2-1)\,dz - z\,d(x^2+y^2-1)}{(x^2+y^2-1)^2+z^2}$$

$$\int_K \omega_L = 1. \qquad K: \ x=0,\ y=0$$

$$\omega_L = \frac{1}{2\pi} \frac{(x^2+y^2-1)\,dz - z\,(2x\,dx + 2y\,dy)}{(x^2+y^2-1)^2+z^2}$$

$$\int_{K:\, x=0,\, y=\omega} \omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{-dz}{1+z^2} = -\frac{\tan^{-1} z}{2\pi}\Big|_{-\infty}^{+\infty}$$

$$= \frac{-\left(\frac{\pi}{2}+\frac{\pi}{2}\right)}{2\pi} = -1/2.$$



$$K = \lim_{R\to\infty} C_R$$

$$\lim_{R\to\infty}\int_{C_R} \omega_L = 1$$

Mayer-Vietoris Sequence

Chain : (A_*, d_*)

$$\rightarrow A_{n-1} \xrightarrow{d_{n-1}} A_n \xrightarrow{d_n} A_{n+1} \xrightarrow{d_{n+1}} \cdots$$

A_n: $\mathbb{R}$ vector space, d_n Homomorphism such that $d_n \circ d_{n-1} = 0$ for all n.

$\operatorname{Im} d_{n-1} \subseteq \ker d_n$

$$H^n(A_*, d_*) \doteq \frac{\ker d_n}{\operatorname{Im} d_{n-1}}.$$

Now consider homomorphism of chain complexes.

$$\cdots \rightarrow A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \rightarrow \cdots$$

$$\cdots \rightarrow A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \rightarrow \cdots$$

The sequence $0 \rightarrow A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \rightarrow 0$

is called short exact if

1) f_n is injective
2) $\operatorname{Im} f_n = \ker g_n$
3) g_n is onto.

If these are all vector spaces then

$B_n \simeq A_n \oplus C_n$.

Ex $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$

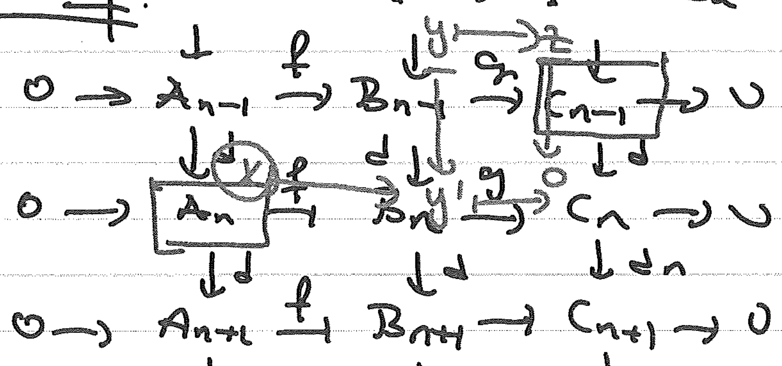
Short exact but $\mathbb{Z}$ is <u>not</u> isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$.

<u>Theorem:</u> Suppose $0 \to A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \to 0$ is a short exact sequence of chain complexes. Then there is a long exact sequence of the form

$$\cdots \xrightarrow{\delta} H^n(A_*) \xrightarrow{f^*} H^n(B_*) \xrightarrow{g^*} H^n(C_*) \xrightarrow{\delta} H^{n+1}(A_*) \xrightarrow{f^*} \cdots$$

where δ is called the connecting homomorphism.

<u>Proof:</u> $0 \to A_* \to B_* \to C_* \to 0$ means that



$H^n(A) \xrightarrow{f^*} H^n(B)$: $\delta : H^{n-1}(C_*) \to H^n(A)$

$[z] \in H^{n-1}(C_*)$, $\delta([z]) \in H^n(A)$ (with "?" written above the equals sign)

$dz = 0$ $\qquad \delta([z]) = [x]$ well defined!

We'll use this machinery in the following form.

Let M be a smooth manifold and assume $M = U \cup V$, where U, V are open subsets.

Then we have a short exact sequence of chain complexes:

$$U \cap V \xrightarrow{j_U} U \xrightarrow{i_U} M, \qquad U \cap V \xrightarrow{j_V} V \xrightarrow{i_V} M \qquad \text{inclusion maps}$$

$$\Omega^k(U \cap V) \xleftarrow{j_U^*} \Omega^k(U) \xleftarrow{i_U^*} \Omega^k(M), \qquad \Omega^k(U \cap V) \xleftarrow{j_V^*} \Omega^k(V) \xleftarrow{i_V^*} \Omega^k(M)$$

Consider the following sequence:

$$\Omega^k(M) \xrightarrow[f_k]{(i_U^*,\, i_V^*)} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{g_k} \Omega^k(U \cap V)$$

$A_k = \Omega^k(M)$, $B_k = \Omega^k(U) \oplus \Omega^k(V)$, $C_k = \Omega^k(U \cap V)$, $g_k = j_U^* - j_V^*$

$$\omega \longmapsto (\omega|_U, \omega|_V) \longmapsto \omega|_{U \cap V} - \omega|_{U \cap V} = 0$$

<u>Claim</u> The sequence $0 \to A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \to 0$ is short exact.

(Exercise: Prove the claim)

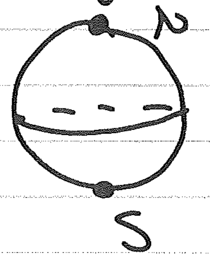
<u>Applications</u>: The above short exact sequence of cochains gives rise to the following long exact sequence: (Mayer-Vietoris exact seq)

$$\xrightarrow{\delta} H^k_{DR}(M) \to H^k_{DR}(U) \oplus H^k_{DR}(V) \to H^k_{DR}(U \cap V) \xrightarrow{\delta} H^{k+1}_{DR}(M)$$

$$[\omega] \longmapsto ([\omega|_U], [\omega|_V])$$

$$([\nu], [\eta]) \mapsto [\nu - \eta]$$

<u>Example</u>: <u>Claim</u>: $H^k_{DR}(S^n) = \begin{cases} \mathbb{R} & \text{if } k = 0 \text{ or } n \\ 0 & \text{otherwise.} \end{cases}$



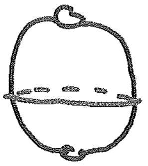
$N = (0, \dots, 1)$, $S = (0, \dots, -1)$

$U = S^n \setminus \{S\}$, $V = S^n \setminus \{N\}$

$U \simeq \mathbb{R}^n \simeq V$ diffeomorphic and thus

$H^k_{DR}(U) \simeq H^k_{DR}(V) = 0$ if $k > 0$ and $\simeq \mathbb{R}$ if $k=0$.

$U \cap V = S^n \setminus \{N, S\} \xleftrightarrow{h.e.} S^{n-1}$ the equator

$U \cap V$  $S^{n-1} \longrightarrow$  $S^{n-1} \overset{h.e.}{\simeq} U \cap V$

So we have the Meyer-Vietoris exact sequence

$$\dashrightarrow H^{k-1}_{DR}(S^n) \to H^{k-1}_{DR}(U) \oplus H^{k-1}_{DR}(V) \to H^{k-1}_{DR}(S^{n-1}) \to$$

First assume $k = 1$.

$$0 \to H^0_{DR}(S^n) \to H^0_{DR}(U) \oplus H^0_{DR}(V) \to H^0_{DR}(S^{n-1}) \to 0$$

$H^0_{DR}(S^n) \cong \mathbb{R}$, the first map is inj., $H^0_{DR}(U) \cong \mathbb{R}$, $H^0_{DR}(V) \cong \mathbb{R}$, the second map is surj., $H^0_{DR}(S^{n-1}) \cong \mathbb{R}$ if $n > 1$

$$[a] \longmapsto ([a], [a])$$

$$([a], [b]) \longmapsto [a-b]$$

$S^0 = \{\pm 1\}$

$\underline{n=1}$ $\quad 0 \to \mathbb{R} \xrightarrow{i^*} \mathbb{R} \oplus \mathbb{R} \xrightarrow{j^*} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\delta} H^1_{DR}(S^1) \to 0$

$$a \longmapsto ([a], [a])$$

$$([a], [b]) \longmapsto ([a-b], [a-b])$$

$$\mathbb{R} \simeq \operatorname{Im} j = \{([x], [x]) \mid x\} = \ker \delta$$

$$H^1_{DR}(S^1) \simeq \mathbb{R}\oplus\mathbb{R} / \ker\delta \simeq \frac{\mathbb{R}\oplus\mathbb{R}}{\mathbb{R}} \simeq \mathbb{R}.$$

Assume now $n > 1$.

$1 < k < n$

$$\dashrightarrow \underbrace{H^{k-1}_{DR}(U) \oplus H^{k-1}_{DR}(V)}_{\simeq 0} \to H^{k-1}_{DR}(S^{n-1}) \xrightarrow{\delta} H^k_{DR}(S^n) \to$$

$$= 0$$

$$0 \to H^{k-1}_{DR}(S^{n-1}) \xrightarrow{\delta} H^k_{DR}(S^n) \to 0$$

<u>k=n</u> $H^k(S^n) \cong H^1(S^1) = \mathbb{R}$.

<u>Exercise</u>: Fill the details.

<u>Example</u>: $H^k_{DR}(\mathbb{CP}^n) = \begin{cases} \mathbb{R} & k = 0, 2, \ldots, 2n \\ 0 & \text{otherwise.} \end{cases}$

We know that $H^k_{DR}(\mathbb{CP}^1) = H^k_{DR}(S^2) = \begin{cases} \mathbb{R} & k = 0, 2 \\ 0 & \text{other} \end{cases}$